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R-GROUPS AND PARAMETERS

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Let G be a p-adic group, SO_{2n+1}, Sp_{2n}, O_{2n} or U_n. Let π be an irreducible discrete series representation of a Levi subgroup of G. We prove the conjecture that the Knapp–Stein R-group of π and the Arthur R-group of π are isomorphic. Several instances of the conjecture were established earlier: for archimedean groups by Shelstad; for principal series representations by Keys; for $G = \mathrm{SO}_{2n+1}$ by Ban and Zhang; and for $G = \mathrm{SO}_n$ or Sp_{2n} in the case when π is supercuspidal, under an assumption on the parameter, by Goldberg.

1. Introduction

Central to representation theory of reductive groups over local fields is the study of parabolically induced representations. In order to classify the tempered spectrum of such a group, one must understand the structure of parabolically induced from discrete series representations, in terms of components, multiplicities, and whether or not components are elliptic. The Knapp–Stein R-group gives an explicit combinatorial method for conducting this study. On the other hand, the local Langlands conjecture predicts the parametrization of such nondiscrete tempered representations, in L-packets, by admissible homomorphisms of the Weil–Deligne group which factor through a Levi component of the Langlands dual group. Arthur [1989] gave a conjectural description of the Knapp–Stein R-group in terms of the parameter. This conjecture generalizes results of Shelstad [1982] for archimedean groups, as well as those of Keys [1987] in the case of unitary principal series of certain p-adic groups. In [Ban and Zhang 2005] this conjecture was established for odd special orthogonal groups. In [Goldberg 2011] the conjecture was established for induced from supercuspidal representations of split special orthogonal or symplectic groups, under an assumption on the parameter. In the current work, we complete the conjecture for the full tempered spectrum of all these groups.

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Let F be a nonarchimedean local field of characteristic zero. We denote by $\boldsymbol{G}$ a connected reductive quasi-split algebraic group defined over F. We let $G = \boldsymbol{G}(F)$, and use similar notation for other groups defined over F. Fix a maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$, and a Borel subgroup $\boldsymbol{B} = \boldsymbol{T}\boldsymbol{U}$ containing $\boldsymbol{T}$. We let $\mathscr{E}(G)$ be the equivalence classes of irreducible admissible representations of G, $\mathscr{E}_t(G)$ the tempered classes, $\mathscr{E}_2(G)$ the discrete series, and ${}^\circ\mathscr{E}(G)$ the irreducible unitary supercuspidal classes. We make no distinction between a representation π and its equivalence class.

Let $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{N}$ be a standard, with respect to $\boldsymbol{B}$, parabolic subgroup of $\boldsymbol{G}$. Let $\boldsymbol{A} = \boldsymbol{A}_{\boldsymbol{M}}$ be the split component of $\boldsymbol{M}$, and let $W = W(\boldsymbol{G}, \boldsymbol{A}) = N_{\boldsymbol{G}}(\boldsymbol{A})/\boldsymbol{M}$ be the Weyl group for this situation. For $\sigma \in \mathscr{E}(M)$ we let $\operatorname{Ind}_P^G(\sigma)$ be the representation unitarily induced from $\sigma \otimes \mathbf{1}_N$. Thus, if V is the space of σ, we let

$$V(\sigma) = \left\{ f \in C^\infty(G, V) \mid f(mng) = \delta_P(m)^{1/2} f(g) \text{ for all } m \in M,\ n \in N,\ g \in G \right\},$$

with δ_P the modulus character of P. The action of G is by the right regular representation, so $(\operatorname{Ind}_P^G(\sigma)(x)f)(g) = f(gx)$. Then any $\pi \in \mathscr{E}_t(G)$ is an irreducible component of $\operatorname{Ind}_P^G(\sigma)$ for some choice of M and $\sigma \in \mathscr{E}_2(M)$. In order to determine the component structure of $\operatorname{Ind}_P^G(\sigma)$, Knapp and Stein, in the archimedean case, and Harish-Chandra in the p-adic case, developed the theory of singular integral intertwining operators, leading to the theory of R-groups, due to Knapp and Stein [1971] in the archimedean case and Silberger [1978; 1979] in the p-adic case. We describe this briefly and refer the reader to the introduction of [Goldberg 1994] for more details. The poles of the intertwining operators give rise to the zeros of Plancherel measures. Let $\Phi(\boldsymbol{P}, \boldsymbol{A})$ be the reduced roots of $\boldsymbol{A}$ in $\boldsymbol{P}$. For $\alpha \in \Phi(\boldsymbol{P}, \boldsymbol{A})$ and $\sigma \in \mathscr{E}_2(M)$ we let $\mu_\alpha(\sigma)$ be the rank one Plancherel measure associated to σ and α. We let $\Delta' = \{\alpha \in \Phi(\boldsymbol{P}, \boldsymbol{A}) \mid \mu_\alpha(\sigma) = 0\}$. For $w \in W$ and $\sigma \in \mathscr{E}_2(M)$ we let $w\sigma(m) = \sigma(w^{-1}m\sigma)$. (Note, we make no distinction between $w \in W$ and its representative in $N_G(A)$.) We let

$$W(\sigma) = \{w \in W \mid w\sigma \simeq \sigma\},$$

and let W' be the subgroup of $W(\sigma)$ generated by those w_α with $\alpha \in \Delta'$. We let $R(\sigma) = \{w \in W(\sigma) \mid w\Delta' = \Delta'\} = \{w \in W(\sigma) \mid w\alpha > 0 \text{ for all } \alpha \in \Delta'\}$. Let $\mathscr{C}(\sigma) = \operatorname{End}_G(\operatorname{Ind}_P^G(\sigma))$.

Theorem 1 [Knapp and Stein 1971; Silberger 1978; 1979]. *For any $\sigma \in \mathscr{E}_2(M)$, we have $W(\sigma) = R(\sigma) \ltimes W'$, and $\mathscr{C}(\sigma) \simeq \mathbb{C}[R(\sigma)]_\eta$, the group algebra of $R(\sigma)$ twisted by a certain 2-cocycle η.*

Thus $R(\sigma)$, along with η, determines how many inequivalent components appear in $\operatorname{Ind}_P^G(\sigma)$ and the multiplicity with which each one appears. Furthermore Arthur shows $\mathbb{C}[R(\sigma)]_\eta$ also determines whether or not components of $\operatorname{Ind}_P^G(\sigma)$ are elliptic (and hence whether or not they contribute to the Plancherel formula) [Arthur 1993].

Arthur [1989] conjectured a construction of $R(\sigma)$ in terms of the local Langlands conjecture. Let W_F be the Weil group of F and $W'_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil–Deligne group. Suppose $\psi : W'_F \to {}^L M$ parametrizes the L-packet, $\Pi_\psi(M)$, of M containing σ. Here ${}^L M = \hat{M} \rtimes W_F$ is the Langlands L-group, and $\hat{M}$ is the complex group whose root datum is dual to that of $\boldsymbol{M}$. Then

$$\psi : W'_F \to {}^L M \hookrightarrow {}^L G$$

must be a parameter for an L-packet $\Pi_\psi(G)$ of G. The expectation is that $\Pi_\psi(G)$ consists of all irreducible components of $\mathrm{Ind}_P^G(\sigma')$ for all $\sigma' \in \Pi_\psi(M)$. We let $S_\psi = Z_{\hat{G}}(\mathrm{Im}\,\psi)$, and take S_ψ° to be the connected component of the identity. Let T_ψ be a maximal torus in S_ψ°. Set $W_\psi = W(S_\psi, T_\psi)$, and $W_\psi^\circ = W(S_\psi^\circ, T_\psi)$. Then $R_\psi = W_\psi / W_\psi^\circ$ is called the R-group of the packet $\Pi_\psi(G)$. By duality we can identify W_ψ with a subgroup of W. With this identification, we let $W_{\psi,\sigma} = W_\psi \cap W(\sigma)$ and $W_{\psi,\sigma}^\circ = W_\psi^\circ \cap W(\sigma)$. We then set

$$R_{\psi,\sigma} = W_{\psi,\sigma} / W_{\psi,\sigma}^\circ.$$

We call $R_{\psi,\sigma}$ the Arthur R-group attached to ψ and σ.

Conjecture 2. *For any $\sigma \in \mathscr{E}_2(M)$, we have $R(\sigma) \simeq R_{\psi,\sigma}$.*

In [Ban and Zhang 2005], the first named author and Zhang proved this conjecture in the case $\boldsymbol{G} = \mathrm{SO}_{2n+1}$. In [Goldberg 2011] the second named author confirmed the conjecture when σ is supercuspidal, and $\boldsymbol{G} = \mathrm{SO}_n$ *or* Sp_{2n}, with a mild assumption on the parameter ψ. Here, we complete the proof of the conjecture for Sp_{2n}, or O_n, under assumptions given in Section 2.3.

This work is based on the classification of discrete series for classical p-adic groups of Mœglin and Tadić [2002], and on the results of Mœglin [2002; 2007b]. Subsequent to our submission, Arthur's unfinished book has become available in preprint form [Arthur 2011]. In this long awaited and impressive work, he uses the trace formula to classify the automorphic representations of special orthogonal and symplectic groups in terms of those of $\mathrm{GL}(n)$. An important ingredient in this work is a formulation of the classification at the local places. The results for irreducible tempered representations are obtained from the classification of discrete series using R-groups. Our result on isomorphism of R-groups and their dual version for $\mathrm{SO}(2n+1, F)$ and $\mathrm{Sp}(2n, F)$ (see Theorem 7) also appear in Arthur's work [2011, page 346]. Arthur's proof differs significantly from the one we use here. We work with a rather concrete description of parameters based on Jordan blocks and L-functions, while Arthur works in the general context of his theory.

We now describe the contents of the paper in more detail. In Section 2 we introduce our notation and discuss the classification of $\mathscr{E}_2(M)$ for our groups, due to Mœglin and Tadić, as well as preliminaries on Knapp–Stein and Arthur R-groups.

In Section 3 we consider the parameters ψ and compute their centralizers. In Section 4 we turn to the case of $\boldsymbol{G} = O_{2n}$. Here we show the Arthur R-group agrees with the generalization of the Knapp–Stein R-group as discussed in [Goldberg and Herb 1997]. In Section 5 we complete the proof of the theorem for the induced from discrete series representations of Sp_{2n}, SO_{2n+1}, or O_{2n}.

In Section 6, we study R-groups for unitary groups. These groups are interesting for us because they are not split and the action of the Weil group on the dual group is nontrivial. In addition, the classification of discrete series and description of L-parameters is completed [Mœglin 2007b].

The techniques used here can be used for other groups. In particular we should be able to carry out this process for similitude groups and G_2. Furthermore, the techniques of computing the Arthur R-groups will apply to $GSpin$ groups, as well, and may shed light on the Knapp–Stein R-groups in this case. We leave all of this for future work.

2. Preliminaries

2.1. ***Notation.*** Let F be a nonarchimedean local field of characteristic zero. Let G_n, $n \in \mathbb{Z}^+$, be $\mathrm{Sp}(2n, F)$, $\mathrm{SO}(2n+1, F)$ or $\mathrm{SO}(2n, F)$. We define G_0 to be the trivial group. For $G = G_n$ or $G = \mathrm{GL}(n, F)$, fix the minimal parabolic subgroup consisting of all upper triangular matrices in G and the maximal torus consisting of all diagonal matrices in G. If δ_1, δ_2 are smooth representations of $\mathrm{GL}(m, F)$, $\mathrm{GL}(n, F)$, respectively, we define

$$\delta_1 \times \delta_2 = \mathrm{Ind}_P^G(\delta_1 \otimes \delta_2)$$

where $G = \mathrm{GL}(m+n, F)$ and $P = MU$ is the standard parabolic subgroup of G with Levi factor $M \cong \mathrm{GL}(m, F) \times \mathrm{GL}(n, F)$. Similarly, if δ is a smooth representation of $\mathrm{GL}(m, F)$ and σ is a smooth representation of G_n, we define

$$\delta \rtimes \sigma = \mathrm{Ind}_P^{G_{m+n}}(\delta \otimes \sigma)$$

where $P = MU$ is the standard parabolic subgroup of G_{m+n} with Levi factor $M \cong \mathrm{GL}(m, F) \times G_n$. We denote by $\mathscr{E}_2(G)$ the set of equivalence classes of irreducible square integrable representations of G and by ${}^0\mathscr{E}(G)$ the set of equivalence classes of irreducible unitary supercuspidal representations of G.

We say that a homomorphism $h : X \to \mathrm{GL}(d, \mathbb{C})$ is symplectic (respectively, orthogonal) if h fixes an alternating form (respectively, a symmetric form) on $\mathrm{GL}(d, \mathbb{C})$. We denote by S_a the standard a-dimensional irreducible algebraic representation of $\mathrm{SL}(2, \mathbb{C})$. Then

$$S_a \text{ is } \begin{cases} \text{orthogonal} & \text{for } a \text{ odd}, \\ \text{symplectic} & \text{for } a \text{ even}. \end{cases} \tag{1}$$

Let ρ be an irreducible supercuspidal unitary representation of $\mathrm{GL}(d, F)$. According to the local Langlands correspondence for GL_d [Harris and Taylor 2001; Henniart 2000], attached to ρ is an L-parameter $\varphi : W_F \to \mathrm{GL}(d, \mathbb{C})$. Suppose $\rho \cong \tilde{\rho}$. Then $\varphi \cong \tilde{\varphi}$ and one of the Artin L-functions $L(s, \mathrm{Sym}^2\varphi)$ or $L(s, \bigwedge^2\varphi)$ has a pole. The L-function $L(s, \mathrm{Sym}^2\varphi)$ has a pole if and only if φ is orthogonal. The L-function $L(s, \bigwedge^2\varphi)$ has a pole if and only if φ is symplectic. From [Henniart 2010] we know

(2) $$L(s, \textstyle\bigwedge^2\varphi) = L(s, \rho, \textstyle\bigwedge^2), \text{ and } L(s, \mathrm{Sym}^2\varphi) = \mathrm{L}(s, \rho, \mathrm{Sym}^2),$$

where $L(s, \rho, \bigwedge^2)$ and $L(s, \rho, \mathrm{Sym}^2)$ are the Langlands L-functions as defined in [Shahidi 1981].

Let ρ be an irreducible supercuspidal unitary representation of $\mathrm{GL}(d, F)$ and $a \in \mathbb{Z}^+$. We define $\delta(\rho, a)$ to be the unique irreducible subrepresentation of

$$\rho\|^{(a-1)/2} \times \rho\|^{(a-3)/2} \times \cdots \times \rho\|^{(-(a-1))/2};$$

see [Zelevinsky 1980].

2.2. ***Jordan blocks.*** We now review the definition of Jordan blocks from [Mœglin and Tadić 2002]. Let G be $\mathrm{Sp}(2n, F)$, $\mathrm{SO}(2n+1, F)$ or $O(2n, F)$. For $d \in \mathbb{N}$, let r_d denote the standard representation of $\mathrm{GL}(d, \mathbb{C})$. Define

$$R_d = \begin{cases} \bigwedge^2 r_d & \text{for } G = \mathrm{Sp}(2n, F), O(2n, F), \\ \mathrm{Sym}^2 r_d & \text{for } G = \mathrm{SO}(2n+1, F). \end{cases}$$

Let σ be an irreducible discrete series representation of G_n. Denote by $\mathrm{Jord}(\sigma)$ the set of pairs (ρ, a), where $\rho \in {}^0\mathscr{E}(\mathrm{GL}(d_\rho, F))$, $\rho \cong \tilde{\rho}$, and $a \in \mathbb{Z}^+$, such that

(J-1) a is even if $L(s, \rho, R_{d_\rho})$ has a pole at $s = 0$ and odd otherwise,

(J-2) $\delta(\rho, a) \rtimes \sigma$ is irreducible.

For $\rho \in {}^0\mathscr{E}(\mathrm{GL}(d_\rho, F))$, $\rho \cong \tilde{\rho}$, define

$$\mathrm{Jord}_\rho(\sigma) = \{a \mid (\rho, a) \in \mathrm{Jord}(\sigma)\}.$$

Let $\hat{G}$ denote the complex dual group of G. Then $\hat{G} = \mathrm{SO}(2n+1, \mathbb{C})$ for $G = \mathrm{Sp}(2n, F)$, $\hat{G} = \mathrm{Sp}(2n, \mathbb{C})$ for $G = \mathrm{SO}(2n+1, F)$ and $\hat{G} = O(2n, \mathbb{C})$ for $G = O(2n, F)$.

Lemma 3. *Let σ be an irreducible discrete series representation of G_n. Let ρ be an irreducible supercuspidal self-dual representation of $\mathrm{GL}(d_\rho, F)$ and $a \in \mathbb{Z}^+$. Then $(\rho, a) \in \mathrm{Jord}(\sigma)$ if and only if the following conditions hold*:

(J-1′) *$\rho \otimes S_a$ is of the same type as $\hat{G}$,*

(J-2) *$\delta(\rho, a) \rtimes \sigma$ is irreducible.*

Proof. We will prove that (J-1) ⇔ (J-1′). We know from [Shahidi 1990] that one and only one of the two L-functions $L(s,\rho,\bigwedge^2)$ and $L(s,\rho,\mathrm{Sym}^2)$ has a pole at $s=0$. Suppose $G=\mathrm{Sp}(2n,F)$ or $O(2n,F)$. We consider $L(s,\rho,\bigwedge^2)$. It has a pole at $s=0$ if and only if the parameter $\rho : W_F \to \mathrm{GL}(d_\rho,\mathbb{C})$ is symplectic. According to (1), this is equivalent to $\rho\otimes S_a$ being orthogonal for a even. Therefore, for $(\rho,a)\in\mathrm{Jord}(\sigma)$, a is even if and only if $\rho\otimes S_a$ is orthogonal. For $G=\mathrm{SO}(2n+1,F)$, the arguments are similar. □

2.3. *Assumptions.* In this paper, we use the classification of discrete series for classical p-adic groups of Mœglin and Tadić [Mœglin and Tadić 2002], so we have to make the same assumptions as there. Let σ be an irreducible supercuspidal representation of G_n and let ρ be an irreducible self-dual supercuspidal representation of a general linear group. We make the following assumption:

(BA) $\nu^{\pm(a+1)/2}\rho\rtimes\sigma$ *reduces for*

$$a=\begin{cases}\max \mathrm{Jord}_\rho(\sigma) & \textit{if } \mathrm{Jord}_\rho(\sigma)\neq\varnothing,\\ \quad 0 & \textit{if } L(s,\rho,R_{d_\rho}) \textit{ has a pole at } s=0 \textit{ and } \mathrm{Jord}_\rho(\sigma)=\varnothing,\\ -1 & \textit{otherwise}.\end{cases}$$

Moreover, there are no other reducibility points in $\mathbb{R}$.

In addition, we assume that the L-parameter of σ is given by

$$\varphi_\sigma=\bigoplus_{(\rho,a)\in\mathrm{Jord}(\sigma)}\varphi_\rho\otimes S_a. \tag{3}$$

Here, φ_ρ denotes the L-parameter of ρ given in [Harris and Taylor 2001; Henniart 2000].

Mœglin [2007a], assuming certain Fundamental Lemmas, proved the validity of the assumptions for $\mathrm{SO}(2n+1,F)$ and showed how Arthur's results imply the Langlands classification of discrete series for $\mathrm{SO}(2n+1,F)$.

2.4. *The Arthur R-group.* Let ${}^LG=\hat{G}\rtimes W_F$ be the L-group of G, and suppose ${}^LM$ is the L-group of a Levi subgroup, M, of G. Then ${}^LM$ is a Levi subgroup of ${}^LG$ (see [Borel 1979, Section 3] for the definition of parabolic subgroups and Levi subgroups of ${}^LG$). Suppose ψ is an A-parameter of G which factors through ${}^LM$,

$$\psi : W_F\times\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\longrightarrow {}^LM\subset{}^LG.$$

Then we can regard ψ as an A-parameter of M. Suppose, in addition, the image of ψ is not contained in a smaller Levi subgroup (i.e., ψ is an elliptic parameter of M).

Let S_ψ be the centralizer in $\hat{G}$ of the image of ψ and S_ψ^0 its identity component. If T_ψ is a maximal torus of S_ψ^0, define

$$W_\psi = N_{S_\psi}(T_\psi)/Z_{S_\psi}(T_\psi), \quad W_\psi^0 = N_{S_\psi^0}(T_\psi)/Z_{S_\psi^0}(T_\psi), \quad R_\psi = W_\psi/W_\psi^0.$$

Lemma 2.3 of [Ban and Zhang 2005] and the discussion on page 326 of [Ban and Zhang 2005] imply that W_ψ can be identified with a subgroup of $W(G, A)$.

Let σ be an irreducible unitary representation of M. Assume σ belongs to the A-packet $\Pi_\psi(M)$. If $W(\sigma) = \{w \in W(G, A) \mid w\sigma \cong \sigma\}$, we let

$$W_{\psi,\sigma} = W_\psi \cap W(\sigma), \quad W_{\psi,\sigma}^0 = W_\psi^0 \cap W(\sigma),$$

and take $R_{\psi,\sigma} = W_{\psi,\sigma}/W_{\psi,\sigma}^0$ as the Arthur R-group.

3. Centralizers

Let G be $\mathrm{Sp}(2n, F)$, $\mathrm{SO}(2n+1, F)$ or $O(2n, F)$. Let $\hat{G}$ be the complex dual group of G. Let

$$\psi : W_F \times \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow \hat{G} \subset \mathrm{GL}(N, \mathbb{C})$$

be an A-parameter. We consider ψ as a representation. Then ψ is a direct sum of irreducible subrepresentations. Let ψ_0 be an irreducible subrepresentation. For $m \in \mathbb{N}$, set

$$m\psi_0 = \underbrace{\psi_0 \oplus \cdots \oplus \psi_0}_{m \text{ times}}.$$

If $\psi_0 \ncong \tilde{\psi}_0$, then it can be shown using the bilinear form on $\hat{G}$ that $\tilde{\psi}_0$ is also a subrepresentation of ψ. Therefore, ψ decomposes into a sum of irreducible subrepresentations

$$\psi = (m_1\psi_1 \oplus m_1\tilde{\psi}_1) \oplus \cdots \oplus (m_k\psi_k \oplus m_k\tilde{\psi}_k) \oplus m_{k+1}\psi_{k+1} \oplus \cdots \oplus m_l\psi_l,$$

where $\psi_i \ncong \psi_j$, $\psi_i \ncong \tilde{\psi}_j$ for $i \neq j$. In addition, $\psi_i \ncong \tilde{\psi}_i$ for $i = 1, \ldots, k$ and $\psi_i \cong \tilde{\psi}_i$ for $i = k+1, \ldots, l$. If $\psi_i \cong \tilde{\psi}_i$, then ψ_i factors through a symplectic or orthogonal group. In this case, if ψ_i is not of the same type as $\hat{G}$, then m_i must be even. This follows again using the bilinear form on $\hat{G}$.

We want to compute S_ψ and W_ψ. First, we consider the case $\psi = m\psi_0$ or $\psi = m\psi_0 \oplus m\tilde{\psi}_0$, where ψ_0 is irreducible. The following lemma is an extension of Proposition 6.5 of [Gross and Prasad 1992]. A part of the proof was communicated to us by Joe Hundley.

Lemma 4. *Let G be* $\mathrm{Sp}(2n, F)$, $\mathrm{SO}(2n+1, F)$ *or* $O(2n, F)$. *Let*

$$\psi_0 : W_F \times \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(d_0, \mathbb{C})$$

be an irreducible parameter.

(i) *Suppose* $\psi_0 \not\cong \tilde{\psi}_0$ *and* $\psi = m\psi_0 \oplus m\tilde{\psi}_0$. *Then* $S_\psi \cong \mathrm{GL}(m, \mathbb{C})$ *and* $R_\psi = 1$.

(ii) *Suppose* $\psi_0 \cong \tilde{\psi}_0$ *and* $\psi = m\psi_0$. *Suppose* ψ_0 *is of the same type as* $\hat{G}$. *Then*

$$R_\psi \cong \begin{cases} \mathbb{Z}_2 & \text{for } m \text{ even,} \\ 1 & \text{for } m \text{ odd.} \end{cases}$$

(iii) *Suppose* $\psi_0 \cong \tilde{\psi}_0$ *and* $\psi = m\psi_0$. *Suppose* ψ_0 *is not of the same type as* $\hat{G}$. *Then* m *is even,* $S_\psi \cong \mathrm{Sp}(m, \mathbb{C})$ *and* $R_\psi = 1$.

Proof. (i) The proof of the statement is the same as in [Gross and Prasad 1992].

(ii) and (iii) Suppose $G = \mathrm{Sp}(2n, F)$ or $\mathrm{SO}(2n+1, F)$. Let V and V_0 denote the spaces of the representations ψ and ψ_0, respectively. Denote by $\langle\, , \rangle$ the ψ-invariant bilinear form on V and by $\langle\, , \rangle_0$ the ψ_0-invariant bilinear form on V_0. There exists an isomorphism $V \to V_0 \oplus \cdots \oplus V_0$. Equivalently, $V \cong W \otimes V_0$, where W is a finite dimensional vector space with trivial $W_F \times \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$-action. The space W can be identified with $\mathrm{Hom}_{W_F \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})}(V_0, V)$. Then the map $W \otimes V_0 \to V$ is

$$l \otimes v \mapsto l(v), \qquad l \in \mathrm{Hom}_{W_F \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})}(V_0, V),\ v \in V_0.$$

We claim there exists a nondegenerate bilinear form $\langle\, , \rangle_W$ on W such that $\langle\, , \rangle = \langle\, , \rangle_W \otimes \langle\, , \rangle_0$ in the sense that

$$\langle l_1 \otimes v_1, l_2 \otimes v_2 \rangle = \langle l_1, l_2 \rangle_W \langle v_1, v_2 \rangle_0 \quad \text{for all } l_1, l_2 \in W,\ v_1, v_2 \in V_0.$$

The key ingredient is Schur's lemma, or rather, the variant thereof stating that every invariant bilinear form on V_0 is a scalar multiple of $\langle\, , \rangle_0$. Given any l_1, l_2 in $\mathrm{Hom}_{W_F \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})}(V_0, V)$,

$$\langle l_1(v_1), l_2(v_2) \rangle$$

is an invariant bilinear form on V_0 and therefore it is equal to $c\langle\, , \rangle_0$, for some constant c. We can define $\langle l_1, l_2 \rangle_W$ by

$$\langle l_1, l_2 \rangle_W = \frac{\langle l_1(v_1), l_2(v_2) \rangle}{\langle v_1, v_2 \rangle_0}$$

because Schur's lemma tells us that the right-hand side is independent of v_1, v_2 in V_0. This proves the claim. Observe that if ψ_0 is not of the same type as ψ, the form $\langle\, , \rangle_W$ is alternating, while in the case when ψ_0 and ψ are of the same type, the form $\langle\, , \rangle_W$ is symmetric.

Now, $\mathrm{Im}\,\psi = \{I_m \otimes g \mid g \in \mathrm{Im}\,\psi_0\}$ and

$$\begin{aligned} Z_{\mathrm{GL}(N,\mathbb{C})}(\mathrm{Im}\,\psi) &= \{g \otimes z \mid g \in \mathrm{GL}(m, \mathbb{C}),\ z \in \{\lambda I_{d_0} \mid \lambda \in \mathbb{C}^\times\}\} \\ &= \{g \otimes I_{d_0} \mid g \in \mathrm{GL}(m, \mathbb{C})\}. \end{aligned}$$

Let us denote by $\mathcal{W}$ the group of matrices in $\mathrm{GL}(W)$ which preserve $\langle\,,\rangle_W$, i.e., $\mathcal{W} = \mathrm{Sp}(m, \mathbb{C})$ if $\langle\,,\rangle_W$ is an alternating form and $\mathcal{W} = O(m, \mathbb{C})$ if $\langle\,,\rangle_W$ is a symmetric form. Then

$$S_\psi = Z_{\mathrm{GL}(N,\mathbb{C})}(\mathrm{Im}\,\psi) \cap \hat{G} = \{g \otimes I_{d_0} \mid g \in \mathcal{W},\ \det(g \otimes I_{d_0}) = 1\}.$$

It follows that in case (iii) we have $S_\psi \cong \mathrm{Sp}(m, \mathbb{C})$, $S_\psi^0 = S_\psi$ and $R_\psi = 1$.

In case (ii), $\mathcal{W} = O(m, \mathbb{C})$. Since $\det(g \otimes I_{d_0}) = (\det g)^{d_0}$, it follows

$$S_\psi \cong \begin{cases} O(m, \mathbb{C}), & d_0 \text{ even}, \\ \mathrm{SO}(m, \mathbb{C}), & d_0 \text{ odd}. \end{cases}$$

In the case $G = \mathrm{SO}(2n+1, F)$, ψ_0 is symplectic and d_0 is even. Then $S_\psi \cong O(m, \mathbb{C})$ and $S_\psi^0 \cong \mathrm{SO}(m, \mathbb{C})$. If m is even, this implies $R_\psi \cong \mathbb{Z}_2$. For m odd, $W_\psi = W_\psi^0$ and $R_\psi = 1$.

In the case $G = \mathrm{Sp}(2n, F)$, we have $\hat{G} = \mathrm{SO}(2n+1, \mathbb{C})$ and $md_0 = 2n+1$. It follows that m and d_0 are both odd. Then $S_\psi \cong \mathrm{SO}(m, \mathbb{C})$, $S_\psi^0 = S_\psi$ and $R_\psi = 1$.

The case $G = O(2n, F)$ is similar, but simpler, because there is no condition on determinant. It follows that $S_\psi \cong O(m, \mathbb{C})$. This implies $R_\psi \cong \mathbb{Z}_2$ for m even and $R_\psi = 1$ for m odd. □

Lemma 5. *Let G be $\mathrm{Sp}(2n, F)$, $\mathrm{SO}(2n+1, F)$ or $O(2n, F)$. Let*

$$\psi : W_F \times \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \hat{G}$$

be an A-parameter. We can write ψ in the form

$$\psi \cong \Big(\bigoplus_{i=1}^{p} (m_i\psi_i \oplus m_i\tilde{\psi}_i)\Big) \oplus \Big(\bigoplus_{i=p+1}^{q} 2m_i\psi_i\Big) \oplus \Big(\bigoplus_{i=q+1}^{r} (2m_i+1)\psi_i\Big) \oplus \Big(\bigoplus_{i=r+1}^{s} 2m_i\psi_i\Big), \tag{4}$$

where ψ_i is irreducible for $i \in \{1, \ldots, s\}$, and

$$\begin{aligned} &\psi_i \not\cong \psi_j,\ \psi_i \not\cong \tilde{\psi}_j && \text{for } i, j \in \{1, \ldots, s\},\ i \neq j, \\ &\psi_i \not\cong \tilde{\psi}_i && \text{for } i \in \{1, \ldots, p\}, \\ &\psi_i \cong \tilde{\psi}_i && \text{for } i \in \{p+1, \ldots, s\}, \\ &\psi_i \text{ not of the same type as } \hat{G} && \text{for } i \in \{p+1, \ldots, q\}, \\ &\psi_i \text{ of the same type as } \hat{G} && \text{for } i \in \{q+1, \ldots, s\}. \end{aligned}$$

Let $d = s - r$. Then

$$R_\psi \cong \mathbb{Z}_2^d.$$

Proof. Set $\Psi_i = m_i\psi_i \oplus m_i\tilde{\psi}_i$ for all $i \in \{1, \dots, p\}$, and $\Psi_i = m_i\psi_i$ for all $i \in \{p+1, \dots, s\}$. Denote by Z_i the centralizer of the image of Ψ_i in the corresponding GL. Then

$$Z_{\mathrm{GL}(N,\mathbb{C})}(\operatorname{Im}\psi) = Z_1 \times \cdots \times Z_s \quad \text{and} \quad S_\psi = Z_{\mathrm{GL}(N,\mathbb{C})}(\operatorname{Im}\psi) \cap \hat{G}.$$

Lemma 4 tells us the factors corresponding to $i \in \{1, \dots, q\}$ do not contribute to R_ψ. In addition, we can see from the proof of Lemma 4 that these factors do not appear in determinant considerations. Therefore, we can consider only the factors corresponding to $i \in \{q+1, \dots, s\}$. Let $\mathcal{Z} = Z_{q+1} \times \cdots \times Z_s$ and $\mathcal{S} = \mathcal{Z} \cap \hat{G}$. In the same way as in the proof of Lemma 4, we obtain

$$\mathcal{S} \cong \Big\{(g_{q+1},\dots,g_s) \mid g_i \in O(2m_i+1,\mathbb{C}),\ i \in \{q+1,\dots,r\}, \\ g_i \in O(2m_i,\mathbb{C}), i \in \{r+1,\dots,s\},\ \prod_{i=q+1}^{s} (\det g_i)^{\dim\psi_i} = 1\Big\}, \tag{5}$$

for $G = \mathrm{SO}(2n+1, F)$ or $\mathrm{Sp}(2n, F)$. For $G = O(2n, F)$, we omit the condition on determinant. If $G = \mathrm{SO}(2n+1, F)$, then for $i \in \{q+1, \dots, s\}$, ψ_i is symplectic and $\dim\psi_i$ is even. Therefore, the product in (5) is always equal to 1.

Now, for $G = \mathrm{SO}(2n+1, F)$ and $G = O(2n, F)$, we have

$$\mathcal{S} \cong \prod_{i=q+1}^{r} O(2m_i+1,\mathbb{C}) \times \prod_{i=r+1}^{s} O(2m_i,\mathbb{C}).$$

It follows that $R_\psi \cong \prod_{i=q+1}^{r} 1 \times \prod_{i=r+1}^{s} \mathbb{Z}_2 \cong \mathbb{Z}_2^d$.

It remains to consider $G = \mathrm{Sp}(2n, F)$, $\hat{G} = \mathrm{SO}(2n+1, \mathbb{C})$. We have

$$\sum_{i=1}^{q} 2m_i \dim\psi_i + \sum_{i=q+1}^{r} (2m_i+1)\dim\psi_i + \sum_{i=1}^{p} 2m_i \dim\psi_i = 2n+1.$$

Since the total sum is odd, we must have $r > q$ and $\dim\psi_i$ odd, for some $i \in \{q+1, \dots, r\}$. Without loss of generality, we may assume $\dim\psi_{q+1}$ odd. Then

$$\mathcal{S} \cong \mathrm{SO}(2m_{q+1}+1,\mathbb{C}) \times \prod_{i=q+2}^{r} O(2m_i+1,\mathbb{C}) \times \prod_{i=r+1}^{s} O(2m_i,\mathbb{C}).$$

It follows $R_\psi \cong 1 \times \prod_{i=q+2}^{r} 1 \times \prod_{i=r+1}^{s} \mathbb{Z}_2 \cong \mathbb{Z}_2^d$. □

4. Even orthogonal groups

4.1. ***R-groups for nonconnected groups.*** In this section, we review some results of [Goldberg and Herb 1997]. Let G be a reductive F-group. Let G^0 be the connected component of the identity in G. We assume that G/G^0 is finite and abelian.

Let π be an irreducible unitary representation of G. We say that π is discrete series if the matrix coefficients of π are square integrable modulo the center of G.

We will consider the parabolic subgroups and the R-groups as defined in [Goldberg and Herb 1997]. Let $P^0 = M^0 U$ be a parabolic subgroup of G^0. Let A be the split component in the center of M^0. Define $M = C_G(A)$ and $P = MU$. Then P is called the cuspidal parabolic subgroup of G lying over P^0. The Lie algebra $\mathscr{L}(G)$ can be decomposed into root spaces with respect to the roots Φ of $\mathscr{L}(A)$,

$$\mathscr{L}(G) = \mathscr{L}(M) \oplus \sum_{\alpha\in\Phi} \mathscr{L}(G)_\alpha.$$

Let σ be an irreducible unitary representation of M. We denote by $r_{M^0,M}(\sigma)$ the restriction of σ to M^0. Then, by Lemma 2.21 of [Goldberg and Herb 1997], σ is discrete series if and only if any irreducible constituent of $r_{M^0,M}(\sigma)$ is discrete series. Now, suppose σ is discrete series. Let σ_0 be an irreducible constituent of $r_{M^0,M}(\sigma)$. Then σ_0 is discrete series and we have the Knapp–Stein R-group $R(\sigma_0)$ for $i_{G^0,M^0}(\sigma_0)$ [Knapp and Stein 1971; Silberger 1978]. We review the definition of $R(\sigma_0)$. Let $W(G^0, A) = N_{G^0}(A)/M^0$ and $W_{G^0}(\sigma_0) = \{w \in W_G(M) \mid w\sigma_0 \cong \sigma_0\}$. For $w \in W_{G^0}(\sigma_0)$, we denote by $\mathscr{A}(w, \sigma_0)$ the normalized standard intertwining operator associated to w (see [Silberger 1979]). Define

$$W^0_{G^0}(\sigma_0) = \{w \in W_{G^0}(\sigma_0) \mid \mathscr{A}(w, \sigma_0) \text{ is a scalar}\}.$$

Then $W^0_{G^0}(\sigma_0) = W(\Phi_1)$ is generated by reflections in a set Φ_1 of reduced roots of (G, A). Let Φ^+ be the positive system of reduced roots of (G, A) determined by P and let $\Phi_1^+ = \Phi_1 \cap \Phi^+$. Then

$$R(\sigma_0) = \{w \in W_{G^0}(\sigma_0) \mid w\beta \in \Phi^+ \text{ for all } \beta \in \Phi_1^+\}$$

and $W_{G^0}(\sigma_0) = R(\sigma_0) \ltimes W(\Phi_1)$.

For the definition of $R(\sigma)$, we follow [Goldberg and Herb 1997]. Define

$$\begin{aligned} N_G(\sigma) &= \{g \in N_G(M) \mid g\sigma \cong \sigma\}, \\ W_G(\sigma) &= N_G(\sigma)/M, \quad \text{and} \\ R(\sigma) &= \{w \in W_G(\sigma) \mid w\beta \in \Phi^+ \text{ for all } \beta \in \Phi_1^+\}. \end{aligned}$$

For $w \in W_G(\sigma)$, let $\mathscr{A}(w, \sigma)$ denote the intertwining operator on $i_{G,M}(\sigma)$ defined in [Goldberg and Herb 1997, page 135]. Then the $\mathscr{A}(w, \sigma)$, $w \in R(\sigma)$, form a basis for the algebra of intertwining operators on $i_{G,M}(\sigma)$, by Theorem 5.16 of [Goldberg and Herb 1997]. In addition, $W_G(\sigma) = R(\sigma) \ltimes W(\Phi_1)$. For $w \in W_G(\sigma)$, $\mathscr{A}(w, \sigma)$ is a scalar if and only if $w \in W(\Phi_1)$; see [Goldberg and Herb 1997, Lemma 5.20].

4.2. ***Even orthogonal groups.*** Let $G = O(2n, F)$ and $G^0 = \mathrm{SO}(2n, F)$. Then $G = G^0 \rtimes \{1, s\}$, where $s = \mathrm{diag}\big(I_{n-1}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), I_{n-1}\big)$ and it acts on G^0 by conjugation.

(a) Let

$$M^0 = \{\mathrm{diag}(g_1, \dots, g_r, h, {}^\tau g_r^{-1}, \dots, {}^\tau g_1^{-1}) \mid g_i \in \mathrm{GL}(n_i, F), h \in \mathrm{SO}(2m, F)\}$$
$$\cong \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_r, F) \times \mathrm{SO}(2m, F),$$

where $m > 1$ and $n_1 + \cdots + n_r + m = n$. Then M^0 is a Levi subgroup of G^0. The split component of M^0 is

$$A = \{\mathrm{diag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r}, I_{2m}, \lambda_r^{-1} I_{n_r}, \dots, \lambda_1^{-1} I_{n_1}) \mid \lambda_i \in F^\times\}.$$

Then $M = C_G(A)$ is equal to

$$M = \{\mathrm{diag}(g_1, \dots, g_r, h, {}^\tau g_r^{-1}, \dots, {}^\tau g_1^{-1}) \mid g_i \in \mathrm{GL}(n_i, F), h \in O(2m, F)\} \tag{6}$$
$$\cong \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_r, F) \times O(2m, F).$$

Let $\pi \in \mathscr{E}_2(M)$. Then $\pi \cong \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$, where $\rho_i \in \mathscr{E}_2(\mathrm{GL}(n_i, F))$ and $\sigma \in \mathscr{E}_2(O(2m, F))$. Let $\pi_0 \cong \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma_0$ be an irreducible component of $r_{M^0,M}(\pi)$. If $s\sigma_0 \cong \sigma_0$, then $W_G(\pi) = W_{G^0}(\pi_0)$ and $R(\pi) = R(\pi_0)$. In this case, $r_{M^0,M}(\pi) = \pi_0$, by Lemma 4.1 of [Ban and Jantzen 2003], and $\rho_i \rtimes \sigma$ is reducible if and only if $\rho_i \rtimes \sigma_0$ is reducible, by Proposition 2.2 of [Goldberg 1995]. Then Theorem 6.5 of [Goldberg 1994] tells us that $R(\pi) \cong \mathbb{Z}_2^d$, where d is the number of inequivalent ρ_i with $\rho_i \rtimes \sigma$ reducible.

Now, consider the case $s\sigma_0 \ncong \sigma_0$. It follows from Lemma 4.1 of [Ban and Jantzen 2003] that $\pi = i_{M,M^0}(\pi_0)$. Then $i_{G,M}(\pi) = i_{G,M^0}(\pi_0)$ and we know from Theorem 3.3 of [Goldberg 1995] that $R(\pi) \cong \mathbb{Z}_2^d$, where $d = d_1 + d_2$, d_1 is the number of inequivalent ρ_i such that n_i is even and $\rho_i \rtimes \sigma$ is reducible, and d_2 is the number of inequivalent ρ_i such that n_i is odd and $\rho_i \cong \tilde{\rho}_i$. Moreover, Corollary 3.4 of [Goldberg 1995] implies if n_i is odd and $\rho_i \cong \tilde{\rho}_i$, then $\rho_i \rtimes \sigma$ is reducible. Therefore, we see that $R(\pi) \cong \mathbb{Z}_2^d$, where d is the number of inequivalent ρ_i with $\rho_i \rtimes \sigma$ reducible.

In the case $m = 1$, since

$$\mathrm{SO}(2, F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in F^\times \right\},$$

we have

$$M^0 = \{\mathrm{diag}(g_1, \dots, g_r, a, a^{-1}, {}^\tau g_r^{-1}, \dots, {}^\tau g_1^{-1}) \mid g_i \in \mathrm{GL}(n_i, F), a \in F^\times\}$$
$$\cong \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_r, F) \times \mathrm{GL}(1, F),$$

and this case is described in (b).

(b) Let M^0 be a Levi subgroup of G^0 of the form

$$M^0 = \{\mathrm{diag}(g_1, \dots, g_r, {}^\tau g_r^{-1}, \dots, {}^\tau g_1^{-1}) \mid g_i \in \mathrm{GL}(n_i, F)\}$$

where $n_1 + \cdots + n_r = n$. The split component of M^0 is

$$A = \{\mathrm{diag}(\lambda_1 I_{n_1}, \ldots, \lambda_r I_{n_r}, \lambda_r^{-1} I_{n_r}, \ldots, \lambda_1^{-1} I_{n_1}) \mid \lambda_i \in F^\times\}$$

and $M = C_G(A) = M^0$. Therefore,

$$\begin{aligned} M &= \{\mathrm{diag}(g_1, \ldots, g_r, {}^\tau g_r^{-1}, \ldots, {}^\tau g_1^{-1}) \mid g_i \in \mathrm{GL}(n_i, F)\} \\ &\cong \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_r, F). \end{aligned} \tag{7}$$

Let $\pi \cong \rho_1 \otimes \cdots \otimes \rho_k \otimes 1 \in \mathscr{E}_2(M)$, where 1 denotes the trivial representation of the trivial group. Since $M = M^0$, we can apply directly Theorem 3.3 of [Goldberg 1995]. It follows $R(\pi) \cong \mathbb{Z}_2^d$, where $d = d_1 + d_2$, d_1 is the number of inequivalent ρ_i such that n_i is even and $\rho_i \rtimes 1$ is reducible, and d_2 is the number of inequivalent ρ_i such that n_i is odd and $\rho_i \cong \tilde{\rho}_i$. As above, it follows from Corollary 3.4 of [Goldberg 1995] that if n_i is odd and $\rho_i \cong \tilde{\rho}_i$, then $\rho_i \rtimes \sigma$ is reducible. Again, we obtain $R(\pi) \cong \mathbb{Z}_2^d$, where d is the number of inequivalent ρ_i with $\rho_i \rtimes \sigma$ reducible.

We summarize the above considerations in the following lemma. Observe that the group $O(2, F)$ does not have square integrable representations. It also does not appear as a factor of cuspidal Levi subgroups of $O(2n, F)$. We call a subgroup M defined by (6) or (7) a standard Levi subgroup of $O(2n, F)$.

Lemma 6. *Let $G = O(2n, F)$ and consider a standard Levi subgroup of G of the form*

$$M \cong \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_r, F) \times O(2m, F),$$

where $m \geq 0$, $m \neq 1$, $n_1 + \cdots + n_r + m = n$. Let $\pi \cong \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma \in \mathscr{E}_2(M)$. Then $R(\pi) \cong \mathbb{Z}_2^d$, where d is the number of inequivalent ρ_i with $\rho_i \rtimes \sigma$ reducible.

5. *R*-groups of discrete series

Let G be $\mathrm{Sp}(2n, F)$, $\mathrm{SO}(2n+1, F)$ or $O(2n, F)$.

Theorem 7. *Let π be an irreducible discrete series representation of a standard Levi subgroup M of G_n. Let φ be the L-parameter of π. Then $R_{\varphi,\pi} \cong R(\pi)$.*

Proof. We can write π in the form

$$\pi \cong (\otimes^{m_1} \delta_1) \otimes \cdots \otimes (\otimes^{m_r} \delta_r) \otimes \sigma \tag{8}$$

where σ is an irreducible discrete series representation of G_m and δ_i $(i = 1, \ldots, r)$ is an irreducible discrete series representation of $\mathrm{GL}(n_i, F)$ such that $\delta_i \ncong \delta_j$ for $i \neq j$. As explained in Section 4, if $G_n = O(2n, F)$, then $m \neq 1$.

Let φ_i denote the L-parameter of δ_i and φ_σ the L-parameter of σ. Then the L-parameter φ of π is

$$\varphi \cong (m_1\varphi_1 \oplus m_1\tilde{\varphi}_1) \oplus \cdots \oplus (m_r\varphi_r \oplus m_r\tilde{\varphi}_r) \oplus \varphi_\sigma.$$

Each φ_i is irreducible. The parameter φ_σ is of the form $\varphi_\sigma = \varphi'_1 \oplus \cdots \oplus \varphi'_s$ where φ'_i are irreducible, $\varphi'_i \cong \tilde{\varphi}'_i$ and $\varphi'_i \not\cong \varphi'_j$ for $i \neq j$. In addition, φ'_i factors through a group of the same type as $\hat{G}_n$. The sets $\{\varphi_i \mid i = 1, \dots, r\}$ and $\{\varphi'_i \mid i = 1, \dots, s\}$ can have nonempty intersection. After rearranging the indices, we can write φ as

$$\varphi \cong \left(\bigoplus_{i=1}^{h} (m_i\varphi_i \oplus m_i\tilde{\varphi}_i) \right) \oplus \left(\bigoplus_{i=h+1}^{q} 2m_i\varphi_i \right) \oplus \left(\bigoplus_{i=q+1}^{k} 2m_i\varphi_i \right) \oplus \left(\bigoplus_{i=k+1}^{r} (2m_i+1)\varphi_i \right) \oplus \left(\bigoplus_{i=r+1}^{l} \varphi_i \right),$$

where $\varphi_\sigma = \bigoplus_{i=k+1}^{l} \varphi_i$ and

$$\begin{aligned} \varphi_i \not\cong \varphi_j,\ \varphi_i \not\cong \tilde{\varphi}_j \quad & \text{for } i, j \in \{1, \dots, l\},\ i \neq j, \\ \varphi_i \not\cong \tilde{\varphi}_i \quad & \text{for } i \in \{1, \dots, h\}, \\ \varphi_i \cong \tilde{\varphi}_i \quad & \text{for } i \in \{h+1, \dots, l\}, \\ \varphi_i \text{ not of the same type as } \hat{G} \quad & \text{for } i \in \{h+1, \dots, q\}, \\ \varphi_i \text{ of the same type as } \hat{G} \quad & \text{for } i \in \{q+1, \dots, k\}. \end{aligned}$$

Let $d = k - q$. Lemma 5 implies $R_\varphi \cong \mathbb{Z}_2^d$. In addition, $R_{\varphi,\pi} \cong R_\varphi$.

On the other hand, we know that $R(\pi) \cong \mathbb{Z}_2^c$, where c is cardinality of the set

$$C = \{i \in \{1, \dots, r\} \mid \delta_i \rtimes \sigma \text{ is reducible}\}.$$

This follows from [Goldberg 1994] for $G = \mathrm{SO}(2n+1, F)$ and $G = \mathrm{Sp}(2n, F)$, and from Lemma 6 for $G = O(2n, F)$. We want to show $C = \{q+1, \dots, k\}$. For any $i \in \{1, \dots, l\}$, φ_i is an irreducible representation of $W_F \times \mathrm{SL}(2, \mathbb{C})$ and therefore it can be written in the form $\varphi_i = \varphi'_i \otimes S_{a_i}$, where φ'_i is an irreducible representation of W_F and S_{a_i} is the standard irreducible a_i-dimensional algebraic representation of $\mathrm{SL}(2, \mathbb{C})$. For $i \in \{1, \dots, r\}$, this parameter corresponds to the representation $\delta(\rho_i, a_i)$. Therefore, the representation δ_i in (8) is $\delta_i = \delta(\rho_i, a_i)$. From (3), we have

$$\varphi_\sigma = \bigoplus_{i=k+1}^{l} \varphi_i = \bigoplus_{(\rho,a) \in \mathrm{Jord}(\sigma)} \varphi_\rho \otimes S_a.$$

For $i \in \{h+1, \dots, q\}$, φ_i is not of the same type as $\hat{G}$ and $\delta(\rho_i, a_i) \rtimes \sigma$ is irreducible. For $i \in \{q+1, \dots, k\}$, φ_i is of the same type as $\hat{G}$. Now, Lemma 3 tells us $(\rho_i, a_i) \in \mathrm{Jord}(\sigma)$ if and only if $\delta(\rho_i, a_i) \rtimes \sigma$ is irreducible. Therefore, $\delta(\rho_i, a_i) \rtimes \sigma$ is irreducible for $i \in \{k+1, \dots, r\}$ and $\delta(\rho_i, a_i) \rtimes \sigma$ is reducible for $i \in \{q+1, \dots, k\}$. It follows $C = \{q+1, \dots, k\}$ and $R(\pi) \cong \mathbb{Z}_2^d \cong R_{\varphi,\pi}$, finishing the proof. □

6. Unitary groups

Let E/F be a quadratic extension of p-adic fields. Fix $\theta \in W_F \setminus W_E$. Let $G = U(n)$ be a unitary group defined with respect to E/F, $U(n) \subset \mathrm{GL}(n, E)$. Let

$$J_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \cdot & & & \\ \cdot & & & \end{pmatrix}.$$

We have

$$^L G = \mathrm{GL}(n, \mathbb{C}) \rtimes W_F,$$

where W_E acts trivially on $\mathrm{GL}(n, \mathbb{C})$ and the action of $w \in W_F \setminus W_E$ on $g \in \mathrm{GL}(n, \mathbb{C})$ is given by $w(g) = J_n\, {}^t g^{-1} J_n^{-1}$.

6.1. ***L-parameters for Levi subgroups.*** Suppose we have a Levi subgroup $M \cong \mathrm{Res}_{E/F}\, \mathrm{GL}_k \times U(l)$. Then

$$^L M^0 = \left\{ \begin{pmatrix} g & & \\ & m & \\ & & h \end{pmatrix} \;\middle|\; g, h \in \mathrm{GL}(k, \mathbb{C}),\, m \in \mathrm{GL}(l, \mathbb{C}) \right\}.$$

Direct computation shows that the action of $w \in W_F \setminus W_E$ on $^L M^0$ is given by

$$w\left(\begin{pmatrix} g & & \\ & m & \\ & & h \end{pmatrix}\right) = \begin{pmatrix} J_k\, {}^t h^{-1} J_k^{-1} & & \\ & J_l\, {}^t m^{-1} J_l^{-1} & \\ & & J_k\, {}^t g^{-1} J_k^{-1} \end{pmatrix}.$$

Let π be a discrete series representation of $\mathrm{GL}(k, E) = (\mathrm{Res}_{E/F}\, \mathrm{GL}_k)(F)$ and τ a discrete series representation of $U(l)$. Let $\varphi_\pi : W_E \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(k, \mathbb{C})$ be the L-parameter of π and $\varphi_\tau : W_F \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(l, \mathbb{C}) \rtimes W_F$ the L-parameter of τ. Write

$$\varphi_\tau(w, x) = (\varphi'_\tau(w, x), w), \qquad w \in W_F,\, x \in \mathrm{SL}(2, \mathbb{C}).$$

According to [Borel 1979, Sections 4, 5 and 8], there exists a unique (up to equivalence) L-parameter $\varphi : W_F \times \mathrm{SL}(2, \mathbb{C}) \to {}^L M$ such that

$$\begin{aligned} \varphi((w, x)) &= (\varphi_\pi(w), *, *, w) && \text{for all } w \in W_E,\, x \in \mathrm{SL}(2, \mathbb{C}), \\ \varphi((w, x)) &= (*, \varphi'_\tau(w, x), *, w) && \text{for all } w \in W_F,\, x \in \mathrm{SL}(2, \mathbb{C}). \end{aligned} \tag{9}$$

We will define a map $\varphi : W_F \times \mathrm{SL}(2, \mathbb{C}) \to {}^L M$ satisfying (9) and show that φ is a homomorphism. Define

$$\varphi((w, x)) = (\varphi_\pi(w, x), \varphi'_\tau(w, x), {}^t\varphi_\pi(\theta w \theta^{-1}, x)^{-1}, w), \qquad w \in W_E,\, x \in \mathrm{SL}(2, \mathbb{C}) \tag{10}$$

and

$$\varphi((\theta, 1)) = (J_k^{-1}, \varphi'_\tau(\theta, 1), {}^t\varphi_\pi(\theta^2, 1)^{-1} J_k, \theta).$$

Note that

$$\begin{aligned}\varphi_\tau(\theta^2, 1) &= (\varphi'_\tau(\theta, 1), \theta)(\varphi'_\tau(\theta, 1), \theta)\\ &= (\varphi'_\tau(\theta, 1), 1)(J_l\, {}^t\varphi'_\tau(\theta, 1)^{-1} J_l^{-1}, \theta^2)\\ &= (\varphi'_\tau(\theta, 1) J_l\, {}^t\varphi'_\tau(\theta, 1)^{-1} J_l^{-1}, \theta^2).\end{aligned}$$

It follows that

$$\varphi'_\tau(\theta, 1) J_l\, {}^t\varphi'_\tau(\theta, 1)^{-1} J_l^{-1} = \varphi'_\tau(\theta^2, 1). \tag{11}$$

Similarly, for $w \in W_E$, $x \in \mathrm{SL}(2, \mathbb{C})$,

$$\begin{aligned}\varphi_\tau(\theta w \theta^{-1}, x) &= \varphi_\tau(\theta, 1)\varphi_\tau(w, x)\varphi_\tau(\theta, 1)^{-1}\\ &= (\varphi'_\tau(\theta, 1), \theta)(\varphi'_\tau(w, x), w)(1, \theta^{-1})(\varphi'_\tau(\theta, 1)^{-1}, 1)\\ &= (\varphi'_\tau(\theta, 1), 1)(J_l\, {}^t\varphi'_\tau(w, x)^{-1} J_l^{-1}, \theta w \theta^{-1})(\varphi'_\tau(\theta, 1)^{-1}, 1)\\ &= (\varphi'_\tau(\theta, 1) J_l\, {}^t\varphi'_\tau(w, x)^{-1} J_l^{-1} \varphi'_\tau(\theta, 1)^{-1}, \theta w \theta^{-1})\end{aligned}$$

and thus

$$\varphi'_\tau(\theta, 1) J_l\, {}^t\varphi'_\tau(w, x)^{-1} J_l^{-1} \varphi'_\tau(\theta, 1)^{-1} = \varphi'_\tau(\theta w \theta^{-1}, x). \tag{12}$$

Now,

$$\begin{aligned}&\varphi(\theta, 1)\varphi(\theta, 1)\\ &\quad = \big(J_k^{-1}, \varphi'_\tau(\theta, 1), {}^t\varphi_\pi(\theta^2, 1)^{-1} J_k, \theta\big)\big(J_k^{-1}, \varphi'_\tau(\theta, 1), {}^t\varphi_\pi(\theta^2, 1)^{-1} J_k, \theta\big)\\ &\quad = \big(J_k^{-1}, \varphi'_\tau(\theta, 1), {}^t\varphi_\pi(\theta^2, 1)^{-1} J_k, 1\big)\big(J_k \varphi_\pi(\theta^2, 1), J_l\, {}^t\varphi'_\tau(\theta, 1)^{-1} J_l^{-1}, J_k^{-1}, \theta^2\big)\\ &\quad = \big(\varphi_\pi(\theta^2, 1), \varphi'_\tau(\theta^2, 1), {}^t\varphi_\pi(\theta^2, 1)^{-1}, \theta^2\big) = \varphi(\theta^2, 1),\end{aligned}$$

using (11) and (10). Further, for $w \in W_E$, $x \in \mathrm{SL}(2, \mathbb{C})$, we have

$$\begin{aligned}&\varphi(\theta, 1)\varphi(w, x)\varphi(\theta, 1)^{-1}\\ &\quad = \big(J_k^{-1}, \varphi'_\tau(\theta, 1), {}^t\varphi_\pi(\theta^2, 1)^{-1} J_k, \theta\big)\big(\varphi_\pi(w, x), \varphi'_\tau(w, x), {}^t\varphi_\pi(\theta w \theta^{-1}, x)^{-1}, w\big)\\ &\qquad\qquad \cdot (1, 1, 1, \theta^{-1})\big(J_k, \varphi'_\tau(\theta, 1)^{-1}, J_k^{-1}\, {}^t\varphi_\pi(\theta^2, 1), 1\big)\\ &\quad = \big(J_k^{-1}, \varphi'_\tau(\theta, 1), {}^t\varphi_\pi(\theta^2, 1)^{-1} J_k, 1\big)\\ &\qquad \cdot \big(J_k \varphi_\pi(\theta w \theta^{-1}, x) J_k^{-1}, J_l\, {}^t\varphi'_\tau(w, x)^{-1} J_l^{-1}, J_k\, {}^t\varphi_\pi(w, x)^{-1} J_k^{-1}, \theta w \theta^{-1}\big)\\ &\qquad\qquad \cdot \big(J_k, \varphi'_\tau(\theta, 1)^{-1}, J_k^{-1}\, {}^t\varphi_\pi(\theta^2, 1), 1\big)\\ &\quad = \big(\varphi_\pi(\theta w \theta^{-1}, x), \varphi'_\tau(\theta w \theta^{-1}, x), {}^t\varphi_\pi(\theta^2 w \theta^{-2}, x)^{-1}, \theta w \theta^{-1}\big)\\ &\quad = \varphi(\theta w \theta^{-1}, x).\end{aligned}$$

Here, we use (12) and $J_k^2 = (J_k^{-1})^2 = (-1)^{k-1}$, so

$$^t\varphi_\pi(\theta^2, 1)^{-1} J_k J_k \, {}^t\varphi_\pi(w, x)^{-1} J_k^{-1} J_k^{-1} \, {}^t\varphi_\pi(\theta^2, 1) = {}^t\varphi_\pi(\theta^2 w \theta^{-2}, x)^{-1}.$$

In conclusion, $\varphi(\theta^2, 1) = \varphi(\theta, 1)^2$ and $\varphi(\theta w \theta^{-1}, x) = \varphi(\theta, 1)\varphi(w, x)\varphi(\theta, 1)^{-1}$. Since φ is clearly multiplicative on $W_E \times \mathrm{SL}(2, \mathbb{C})$, it follows that φ is a homomorphism. Therefore, φ is the L-parameter for $\pi \otimes \tau$.

6.2. *The coefficients* λ_φ. Let $\varphi : W_E \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}_k(\mathbb{C})$ be an irreducible L-parameter. Assume $\varphi \cong {}^t({}^\theta\varphi)^{-1}$. Let X be a nonzero matrix such that

$$^t\varphi(\theta w \theta^{-1}, x)^{-1} = X^{-1}\varphi(w, x)X,$$

for all $w \in W_E$, $x \in \mathrm{SL}(2, \mathbb{C})$. We proceed similarly as in [Mœglin 2002, p. 190]. By taking transpose and inverse,

$$\varphi(\theta w \theta^{-1}, x) = {}^tX \, {}^t\varphi(w, x)^{-1} \, {}^tX^{-1}.$$

Next, we replace w by $\theta w \theta^{-1}$. This gives

$$\varphi(\theta^2, 1)\varphi(w, x)\varphi(\theta^{-2}, 1) = {}^tX \, {}^t\varphi(\theta w \theta^{-1}, x)^{-1} \, {}^tX^{-1} = {}^tX X^{-1}\varphi(w, x)X \, {}^tX^{-1},$$

for all $w \in W_E$, $x \in \mathrm{SL}(2, \mathbb{C})$. Since φ is irreducible, $\varphi(\theta^{-2}, 1)\, {}^tX X^{-1}$ is a constant. Define

$$\lambda_\varphi = \varphi(\theta^{-2}, 1)\, {}^tX X^{-1}. \tag{13}$$

As in [Mœglin 2002], we can show that $\lambda_\varphi = \pm 1$.

Lemma 8. *Let $\varphi : W_E \to \mathrm{GL}_k(\mathbb{C})$ be an irreducible L-parameter such that $\varphi \cong {}^t({}^\theta\varphi)^{-1}$. Let S_a be the standard a-dimensional irreducible algebraic representation of* $\mathrm{SL}(2, \mathbb{C})$. *Then ${}^\theta({}^t(\varphi \otimes S_a)^{-1}) \cong \varphi \otimes S_a$ and*

$$\lambda_{\varphi \otimes S_a} = (-1)^{a+1}\lambda_\varphi.$$

Proof. We know that ${}^tS_a^{-1} \cong S_a$. Let Y be a nonzero matrix such that

$$^tS_a(x)^{-1} = Y^{-1}S_a(x)Y,$$

for all $x \in \mathrm{SL}(2, \mathbb{C})$. Then ${}^tY = Y$ for a odd and ${}^tY = -Y$ for a even. Let X be a nonzero matrix such that

$$^t\varphi(\theta w \theta^{-1})^{-1} = X^{-1}\varphi(w)X,$$

for all $w \in W_E$. We have

$$\begin{aligned} {}^t(\varphi \otimes S_a(\theta w \theta^{-1}, x))^{-1} &= ({}^t\varphi(\theta w \theta^{-1})^{-1}) \otimes ({}^tS_a(x)^{-1}) \\ &= (X^{-1}\varphi(w)X) \otimes (Y^{-1}S_a(x)Y) \\ &= (X \otimes Y)^{-1}(\varphi \otimes S_a(w, x)) \otimes (X \otimes Y). \end{aligned}$$

It follows that ${}^{\theta}({}^{t}(\varphi \otimes S_a)^{-1}) \cong \varphi \otimes S_a$ and

$$\begin{aligned}\lambda_{\varphi\otimes S_a} &= (\varphi \otimes S_a(\theta^{-2}, 1))\,{}^{t}(X \otimes Y)(X \otimes Y)^{-1} \\ &= (\varphi(\theta^{-2})\,{}^{t}XX^{-1}) \otimes ({}^{t}YY^{-1})) = (-1)^{a+1}\lambda_{\varphi}.\end{aligned}$$ □

6.3. *Centralizers.* Let $\varphi : W_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^{L}G$ be an L-parameter. Denote by φ_E the restriction of φ to $W_E \times \mathrm{SL}(2,\mathbb{C})$. Then φ_E is a representation of $W_E \times \mathrm{SL}(2,\mathbb{C})$ on $V = \mathbb{C}^n$. Write φ_E as a sum of irreducible subrepresentations

$$\varphi_E = m_1\varphi_1 \oplus \cdots \oplus m_l\varphi_l,$$

where m_i is the multiplicity of φ_i and $\varphi_i \ncong \varphi_j$ for $i \neq j$. It follows from [Mœglin 2002] that S_φ, the centralizer in $\hat{G}$ of the image of φ, is given by

$$S_\varphi \cong \prod_{i=1}^{l} C(m_i\varphi_i), \tag{14}$$

where

$$C(m_i\varphi_i) = \begin{cases} \mathrm{GL}(m_i,\mathbb{C}) & \text{if } \varphi_i \ncong {}^{\theta}\widetilde{\varphi}_i, \\ O(m_i,\mathbb{C}) & \text{if } \varphi_i \cong {}^{\theta}\widetilde{\varphi}_i,\ \lambda_{\varphi_i} = (-1)^{n-1}, \\ \mathrm{Sp}(m_i,\mathbb{C}) & \text{if } \varphi_i \cong {}^{\theta}\widetilde{\varphi}_i,\ \lambda_{\varphi_i} = (-1)^{n}. \end{cases}$$

6.4. *Coefficients λ_ρ.* Let ${}^{L}M = \mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_k(\mathbb{C}) \rtimes W_F$, where the action of $w \in W_F \setminus W_E$ on $\mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_k(\mathbb{C})$ is given by

$$w(g, h, 1)w^{-1} = (J_n\,{}^{t}h^{-1}J_n^{-1}, J_n\,{}^{t}g^{-1}J_n^{-1}, 1).$$

For $\eta = \pm 1$, we denote by R_η the representation of ${}^{L}M$ on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^k)$ given by

$$\begin{aligned} R_\eta((g, h, 1)) \cdot X &= gXh^{-1}, \\ R_\eta((1, 1, \theta)) \cdot X &= \eta J_k\,{}^{t}X J_k. \end{aligned}$$

Let τ denote the nontrivial element in $Gal(E/F)$. Let ρ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(k, E)$. Assume $\rho \cong {}^{\tau}\tilde{\rho}$. Then precisely one of the two L-functions $L(s, \rho, R_1)$ and $L(s, \rho, R_{-1})$ has a pole at $s = 0$. Denote by λ_ρ the value of η such that $L(s, \rho, R_\eta)$ has a pole at $s = 0$.

Lemma 9. *Assume that ρ is an irreducible unitary supercuspidal representation of $\mathrm{GL}(k, E)$ such that $\rho \cong {}^{\tau}\tilde{\rho}$. Let φ_ρ be the L-parameter of ρ. Then $\lambda_{\varphi_\rho} = \lambda_\rho$.*

Proof. As shown in Section 6.1, the parameter $\varphi : W_F \to {}^{L}M$ corresponding to $\varphi_\rho : W_E \to \mathrm{GL}_k(\mathbb{C})$ is given by

$$\varphi(w) = \left(\begin{pmatrix} \varphi_\rho(w) & \\ & {}^{t}\varphi_\rho(\theta w \theta^{-1})^{-1} \end{pmatrix}, w \right), \tag{15}$$

for $w \in W_E$, and

$$\varphi(\theta) = \left(\begin{pmatrix} J_k^{-1} & \\ & {}^t\varphi_\rho(\theta^2)^{-1} J_k \end{pmatrix}, \theta \right). \tag{16}$$

From [Henniart 2010], we have $L(s, \rho, R_\eta) = L(s, R_\eta \circ \varphi)$. Therefore, $L(s, R_{\lambda_\rho} \circ \varphi)$ has a pole at $s = 0$. Then $R_{\lambda_\rho} \circ \varphi$ contains the trivial representation, so there exists nonzero $X \in M_k(\mathbb{C})$ such that $(R_{\lambda_\rho} \circ \varphi)(w) \cdot X = X$ for all $w \in W_F$. In particular, (15) implies that for $w \in W_E$,

$$\varphi_\rho(w) X\, {}^t\varphi_\rho(\theta w \theta^{-1}) = X$$

so

$$\varphi_\rho(w) X = X\, {}^t\varphi_\rho(\theta w \theta^{-1})^{-1}. \tag{17}$$

Therefore, X is a nonzero intertwining operator between φ_ρ and ${}^t({}^\theta\varphi_\rho)^{-1}$. From (13), we have

$$\varphi_\rho(\theta^{-2})\, {}^tX X^{-1} = \lambda_{\varphi_\rho}. \tag{18}$$

Now, since $(R_{\lambda_\rho} \circ \varphi)(\theta) \cdot X = X$, we have from (16)

$${}^tX\, {}^t\varphi_\rho(\theta^2) = \lambda_\rho X.$$

By transposing and multiplying by X^{-1}, we obtain

$$\varphi_\rho(\theta^2) = \lambda_\rho\, {}^tX X^{-1}.$$

We compare this to (18). It follows $\lambda_{\varphi_\rho} = \lambda_\rho$. □

6.5. *Jordan blocks for unitary groups.* For the unitary group $U(n)$, define

$$R_d = R_\eta, \quad \text{where} \quad \eta = (-1)^n.$$

Let σ be an irreducible discrete series representation of $U(n)$. Denote by $Jord(\sigma)$ the set of pairs (ρ, a), where $\rho \in {}^0\mathscr{E}(\mathrm{GL}(d_\rho, E))$, $\rho \cong {}^\tau\tilde{\rho}$, and $a \in \mathbb{Z}^+$, such that (ρ, a) satisfies properties (J-1) and (J-2) from Section 2.2.

Lemma 10. *Let ρ be an irreducible supercuspidal representation of* $\mathrm{GL}(d, E)$ *such that $\varphi_\rho \cong {}^\theta\widetilde{\varphi}_\rho$, where φ_ρ is the L-parameter for ρ. Then the condition* (J-1) *is equivalent to*

(J-1″) $\lambda_{\varphi_\rho \otimes S_a} = (-1)^{n+1}$.

Proof. The condition (J-1) says that a is even if $L(s,\rho,R_d)$ has a pole at $s=0$ and odd otherwise. Observe that

$$\begin{aligned} L(s,\rho,R_d) \text{ has a pole at } s=0 &\Longleftrightarrow \lambda_{\varphi_\rho} = (-1)^n \\ &\Longleftrightarrow \lambda_{\varphi_\rho\otimes S_a} = (-1)^n(-1)^{a+1} \\ &\Longleftrightarrow \lambda_{\varphi_\rho\otimes S_a} = \begin{cases} (-1)^{n+1} & a \text{ even}, \\ (-1)^n & a \text{ odd}. \end{cases} \end{aligned}$$

From this, it is clear that (J-1) is equivalent to (J-1″). ☐

6.6. *R-groups for unitary groups.*

Lemma 11. *Let σ be an irreducible discrete series representation of $U(n)$ and let $\delta=\delta(\rho,a)$ be an irreducible discrete series representation of* $\mathrm{GL}(l,E)$, $l=da$, $d=\dim(\rho)$. *Let φ_ρ and φ be the L-parameters of ρ and $\pi=\delta\otimes\sigma$, respectively. Then $R_{\varphi,\pi}\cong R(\pi)$.*

Proof. Let φ_σ be the L-parameter of σ. Then

$$\varphi_E \cong \varphi_\rho\otimes S_a \oplus {}^\theta\widetilde{\varphi}_\rho\otimes S_a \oplus (\varphi_\sigma)_E.$$

This is a representation of $W_E\times \mathrm{SL}(2,\mathbb{C})$ on $V=\mathbb{C}^{n+2l}$. Write $(\varphi_\sigma)_E$ as a sum of irreducible components,

$$(\varphi_\sigma)_E = \varphi_1\oplus\cdots\oplus\varphi_m.$$

Each component appears with multiplicity one. The centralizer S_φ is given by (14). If $\varphi_\rho\ncong {}^\theta\widetilde{\varphi}_\rho$, then

$$S_\varphi \cong \mathrm{GL}(1,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})\times\prod_{i=1}^m \mathrm{GL}(1,\mathbb{C}).$$

This implies $R_\varphi=1$. On the other hand, $\delta\rtimes\sigma$ is irreducible, so $R(\pi)=1$. It follows $R_{\varphi,\pi}\cong R(\pi)$.

Now, consider the case $\varphi_\rho\cong{}^\theta\widetilde{\varphi}_\rho$. If $\varphi_\rho\otimes S_a\in\{\varphi_1,\ldots,\varphi_m\}$, then

$$S_\varphi\cong O(3,\mathbb{C})\times\prod_{i=1}^{m-1}\mathrm{GL}(1,\mathbb{C}) \quad\text{and}\quad S_\varphi^0\cong \mathrm{SO}(3,\mathbb{C})\times\prod_{i=1}^{m-1}\mathrm{GL}(1,\mathbb{C}).$$

This gives $W_\varphi=W_\varphi^0$ and $R_\varphi=1$. Since $\varphi_\rho\otimes S_a\in\{\varphi_1,\ldots,\varphi_m\}$, the condition (J-2) implies that $\delta\rtimes\sigma$ is irreducible. Therefore, $R(\pi)=1=R_{\varphi,\pi}$.

It remains to consider the case $\varphi_\rho\cong{}^\theta\widetilde{\varphi}_\rho$ and $\varphi_\rho\otimes S_a\notin\{\varphi_1,\ldots,\varphi_m\}$. Then (ρ,a) does not satisfy (J-1″) or (J-2). Assume first that (ρ,a) does not satisfy

(J-1″). Then $\delta \rtimes \sigma$ is irreducible, so $R(\pi) = 1$. Since (ρ, a) does not satisfy (J-1″), we have $\lambda_{\varphi_\rho \otimes S_a} = (-1)^n = (-1)^{n+2l}$. Then, by (14),

$$S_\varphi \cong \mathrm{Sp}(2, \mathbb{C}) \times \prod_{i=1}^{m} \mathrm{GL}(1, \mathbb{C}).$$

It follows $R_{\varphi,\pi} = 1 = R(\pi)$.

Now, assume that (ρ, a) satisfies (J-1″), but does not satisfy (J-2). Then $\lambda_{\varphi_\rho \otimes S_a} = (-1)^{n-1} = (-1)^{n+2l-1}$, so

$$S_\varphi \cong O(2, \mathbb{C}) \times \prod_{i=1}^{m} \mathrm{GL}(1, \mathbb{C})$$

and $R_{\varphi,\pi} \cong \mathbb{Z}_2$. Since (ρ, a) does not satisfy (J-2), $\delta \rtimes \sigma$ is reducible and hence $R(\pi) \cong \mathbb{Z}_2 \cong R_{\varphi,\pi}$. □

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