JACQUET MODULES AND THE LANGLANDS CLASSIFICATION

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Abstract. In this paper, we consider a standard module in the subrepresentation setting of the Langlands classification. We show that the inducing representation appears with multiplicity one in the corresponding Jacquet module, and in fact, it is the unique subquotient of the Jacquet module with its central exponent. As an application, we show how this may be used to easily deduce a dual Langlands classification–essentially, a generalization of the Zelevinsky classification for general linear groups.

1. Introduction

The Langlands classification is a fundamental result in representation theory and the theory of automorphic forms. It gives a bijective correspondence between irreducible admissible representations of a connected reductive group G and triples of Langlands data. It was proved by Langlands for real groups [L]. The proof for p-adic groups is due to Borel and Wallach [B-W], and Silberger [Sil].

We consider the p-adic case, so let G denote a connected reductive p-adic group. Let (P, ν, τ) be a set of Langlands data in the subrepresentation setting of the Langlands classification. Then $P = MU$ is a standard parabolic subgroup of G, τ is an irreducible tempered representation of M and $\nu \in (\mathfrak{a}_M)^*_-$ (see Section 2 for definitions). Write $\pi = L(P, \nu, \tau)$ for the irreducible representation of G corresponding to (P, ν, τ). Then π is the unique irreducible representation of the corresponding standard module, i.e., the induced representation $i_{G,M}(\exp \nu \otimes \tau)$. We show that the

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(normalized) Jacquet module $r_{M,G}(i_{G,M}(\exp\nu\otimes\tau))$ contains $\exp\nu\otimes\tau$ with multiplicity one and has no other subquotients with central exponent ν. This is a useful result (e.g., Lemma 3.4 of [Jan2] is essentially a special case) which was expected, but for which there seems to be no proof available in the literature. Our main purpose here is to fill that gap.

As an application, we prove a dual version of the Langlands classification, essentially extending the Zelevsinky classification from general linear groups to connected reductive groups (cf. [Z]). An irreducible representation θ with unitary central character is called anti-tempered if it satisfies the Casselman criterion for temperedness, except with the usual inequalites reversed. Equivalently, $\hat{\theta}$–its dual under the involution of [Aub], [Be], [S-S]–is tempered. If π is an irreducible admissible representation of G, there exists a unique triple (Q,μ,θ), with $Q=LU$ a standard parabolic subgroup, $\mu\in(\mathfrak{a}_L)^*_+$ and θ an irreducible anti-tempered representation of L, such that π is equivalent to the unique irreducible subrepresentation of $i_{G,L}(\exp\mu\otimes\theta)$ (Theorem 6.3). The growing role of duality in representation theory and its conjectured relation with the Arthur parameterization convinced the aurthors to include this application, especially as it contains information on the composition series (the existence of a unique irreducible subrepresentation) which is not a simple consequence of duality. We note that this is also essentially a known result for which we do not know of a proof in the literature.

In [K], it was noted that there is a problem with Lemma 5.3 [Sil]. We remark that the main result in this paper can serve as a substitute for Lemma 5.3 in Silberger's proof of the Langlands classification.

Our proof of the Jacquet module result is essentially combinatorial in nature. By a result of [B-Z], [Cas], we can calculate $r_{M,G}\circ i_{G,M}(\exp\nu\otimes\tau)$–or the corresponding exponents which appear. Our argument uses the inequalities that arise from having $\nu\in(\mathfrak{a}_M)^*_-$ and τ tempered (the Casselman criterion) to show that any other exponents which appear in $r_{M,G}\circ i_{G,M}(\exp\nu\otimes\tau)$ are necessarily different. As a technical remark, we note that in order to carry out this analysis, the various exponents which appear must be converted to exponents in $\mathfrak{a}^*$ (the dual of the Lie algebra associated to the maximal split torus of the minimal parabolic subgroup).

For general linear groups, the Langlands classification and the Zelevinsky classification are related by the Zelevinsky involution (cf. [Tad]). The duality of [Aub], [Be], [S-S] generalizes the Zelevinsky involution, and may be used in a similar fashion to construct the dual Langlands data for an irreducible admissible representation from the (ordinary) Langlands data for its dual. One issue arises in this process: the duality of [Aub], [Be], [S-S] is at the Grothendieck group level, so does not preserve composition series. To show that π is the unique irreducible subrepresentation of $i_{G,L}(\exp \mu \otimes \theta)$, we note that duality does imply $\exp \mu \otimes \theta$ is the unique irreducible subquotient of $r_{L,G} \circ i_{G,L}(\exp \mu \otimes \theta)$ having its central exponent; the result then follows from Frobenius reciprocity.

We now briefly discuss the contents of the paper section by section. In Section 2, we introduce notation and review some background results. In Section 3, we prove a technical lemma which describes the action of the Weyl group on certain elements in the dual Lie algebra $\mathfrak{a}^*$. This result, together with a criterion for temperedness proved in Section 4 (a variation of the Casselman criterion), is the basis for proving the uniqueness of central characters and central exponents in Section 5. In Section 6, we apply these result to obtain the dual Langlands classification, and show that for general linear groups, it is essentially the same as the Zelvinsky classification.

2. Preliminaries

In this section, we review some background material and notation which will be used in what follows.

Let F be a p-adic field and G the group of F-points of a connected reductive algebraic group defined over F. Fix a maximal split torus A in G. We denote by $W = W(G, A)$ the Weyl group of G with respect to A. Let $\Phi = \Phi(G, A)$ be the set of roots. Fix a minimal parabolic subgroup P_0 containing A. The choice of P_0 determines the set of simple roots $\Pi \subset \Phi$ and the set of positive roots $\Phi^+ \subset \Phi$. If $\alpha \in \Phi^+$, we write $\alpha > 0$.

Let $P = MU \subset G$ be a standard parabolic subgroup of G. We denote by $\Pi_M \subset \Pi$ the corresponding set of simple roots. Let A_M be the split component of the center of M, $X(M)_F$ the group of F-rational characters of M. Let

$$\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbb{R}) = \mathrm{Hom}(X(A_M)_F, \mathbb{R})$$

be the real Lie algebra of A_M and

$$\mathfrak{a}_M^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} = X(A_M)_F \otimes_{\mathbb{Z}} \mathbb{R}$$

its dual. Set $\mathfrak{a} = \mathfrak{a}_A$, $\mathfrak{a}^* = \mathfrak{a}_A^*$. We let $\iota_M : \mathfrak{a}_M^* \to \mathfrak{a}^*$ (or simply ι) denote the standard embedding and $r_M : \mathfrak{a}^* \to \mathfrak{a}_M^*$ restriction ([Sil], [B-W]). Note that we have $r_M \circ \iota_M = id$.

There is a homomorphism (cf. [H-C]) $H_M \colon M \to \mathfrak{a}_M$ such that $q^{\langle \chi, H_M(m) \rangle} = |\chi(m)|$ for all $m \in M$, $\chi \in X(M)_F$. Given $\nu \in \mathfrak{a}_M^*$, let us write

$$\exp \nu = q^{\langle \nu, H_M(\cdot) \rangle}$$

for the corresponding character.

We denote by $i_{G,M}$ the functor of normalized parabolic induction and by $r_{M,G}$ the normalized Jacquet functor. Let $R(G)$ denote the Grothendieck group of the category of smooth finite length representations of G. The Aubert duality operator D_G is defined on $R(G)$ by

$$D_G = \sum_{M \leq G} (-1)^{|\Pi_M|} i_{G,M} \circ r_{M,G},$$

where the sum runs over the set of all standard Levi subgroups of G. Let π be an irreducible smooth representation of G. We define $\hat{\pi} = \pm D_G(\pi)$, taking the sign $+$ or $-$ so that $\hat{\pi}$ is a positive element in $R(G)$.

Lemma 2.1. *Let π be an irreducible smooth representation of G and χ a character of G. Then $\widehat{\chi \otimes \pi} = \chi \otimes \hat{\pi}$ (where $\chi \otimes \pi$ denotes the representation of G defined by $(\chi \otimes \pi)(g) = \chi(g)\pi(g)$).*

Proof. If $P = MU$ is a standandard parabolic subgroup of G and σ is a smooth representation of M, then Proposition 1.9 of [B-Z] implies

$$i_{G,M}(\chi \otimes \sigma) = \chi \otimes i_{G,M}(\sigma), \quad r_{M,G}(\chi \otimes \pi) = \chi \otimes r_{M,G}(\pi).$$

The lemma now follows from the definition of D_G. □

Let $\Pi(P, A_M) = \{r_M(\alpha) \mid \alpha \in \Pi - \Pi_M\}$ denote the set of simple roots for the pair (P, A_M). We have a W-invariant inner product $\langle \cdot, \cdot \rangle : \mathfrak{a}^* \times \mathfrak{a}^* \to \mathbb{R}$. As in [Sil], identifying $\mathfrak{a}_M^*$ with the subspace $\iota(\mathfrak{a}_M^*) \subset \mathfrak{a}^*$, we set

$$(\mathfrak{a}_M)_+^* = \{x \in \mathfrak{a}_M^* \mid \langle x, \alpha \rangle > 0, \forall \alpha \in \Pi(P, A_M)\},$$

$$(\mathfrak{a}_M)_-^* = \{x \in \mathfrak{a}_M^* \mid \langle x, \alpha \rangle < 0, \forall \alpha \in \Pi(P, A_M)\},$$

$$+\bar{\mathfrak{a}}_M^* = \{x \in \mathfrak{a}_M^* \mid x = \sum_{\alpha \in \Pi(P, A_M)} c_\alpha \alpha, \ c_\alpha \geq 0\}.$$

A set of Langlands data for G is a triple (P, ν, τ) with the following properties: (1) $P = MU$ is a standard parabolic subgroup of G, (2) $\nu \in (\mathfrak{a}_M)_-^*$, and (3) τ is (the equivalence class of) an irreducible tempered representation of M.

We now state the Langlands classification (cf. [B-W],[Sil]):

Theorem 2.2 (The Langlands classification)**.** *Suppose (P, ν, τ) is a set of Langlands data for G. Then the induced representation $i_{G,M}(\exp \nu \otimes \tau)$ has a unique irreducible subrepresentation, which we denote by $L(P, \nu, \tau)$. Conversely, if π is an irreducible admissible representation of G, then there exists a unique (P, ν, τ) as above such that $\pi \cong L(P, \nu, \tau)$.*

This theorem describes the Langlands classification in the subrepresentation setting. It can also be formulated in the quotient setting, in which case $\nu \in (\mathfrak{a}_M)_+^*$. We work in the subrepresentation setting for technical reasons: if $\pi \cong L(P,\nu,\tau)$, then $\exp\nu \otimes \tau \leq r_{M,G}(\pi)$.

3. A combinatorial lemma

In this section, we prove a technical lemma which will play a key role in the proof of Proposition 5.3.

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (the set of simple roots). As in [B-W], chapter IX, set $\mathcal{F} = \sum \mathbb{R}\alpha_i$. Then $\mathfrak{a}^* = \mathfrak{z}^* \oplus \mathcal{F}$, where $\mathfrak{z}^* = \{x \in \mathfrak{a}^* \mid \langle x, \alpha\rangle = 0, \forall \alpha \in \Pi\}$. Define $\beta_1, \ldots, \beta_n \in \mathcal{F}$ by $\langle \beta_i, \alpha_j\rangle = \delta_{ij}$. Then $\mathcal{F} = \sum \mathbb{R}\beta_i$. More generally, if $I \subset \{1, \ldots, n\}$, then $\mathfrak{a}^* = \mathfrak{z}^* + \sum_{i \notin I} \mathbb{R}\beta_i + \sum_{i \in I} \mathbb{R}\alpha_i$. Note that if M is the standard Levi factor with $\Pi_M = \{\alpha_i \,|\, i \in I\}$, then $i_M(\mathfrak{a}_M^*) = \mathfrak{z}^* + \sum_{i \notin I} \mathbb{R}\beta_i$. The set of simple roots Π is a basis of an abstract root system in $\mathcal{F}$. Let

$$\bar{\mathcal{F}}_+ = \bar{\mathfrak{a}}_+^* \cap \mathcal{F} = \{x \in \mathcal{F} \mid \langle x, \alpha\rangle \geq 0, \forall \alpha \in \Pi\}.$$

Lemma 3.1. *Let $x, y \in \bar{\mathcal{F}}_+$ and $w \in W$ with $w \neq 1$. Then $\langle wx, y\rangle \leq \langle x, y\rangle$.*

Proof. That $\langle wx - x, y\rangle \leq 0$ is an immediate consequence of Proposition 18 in section 1.6, chapter 5 [Bou]. □

Let $W^{M,A} = \{w \in W \,|\, w\alpha > 0 \text{ for all } \alpha \in \Pi_M\}$.

Lemma 3.2. *Let $P = MU$ be a standard parabolic subgroup of G. Suppose the simple roots of G are labeled so that $\Pi_M = \{\alpha_{k+1}, \ldots, \alpha_n\}$. Let $x = c_1\beta_1 + \cdots + c_k\beta_k + c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n \in \mathcal{F}$, with $c_1, \ldots, c_k < 0$, $c_{k+1}, \ldots, c_n \geq 0$. Let $w \in W^{M,A}$, $w \neq 1$. If $wx = d_1\beta_1 + \cdots + d_k\beta_k + d_{k+1}\alpha_{k+1} + \cdots + d_n\alpha_n$, then $d_i \neq c_i$, for some $i \in \{1, \ldots, k\}$.*

Proof. Let

$$w = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

be the matrix for the action of w with respect to the basis $\beta_1,\ldots,\beta_k,\alpha_{k+1},\ldots,\alpha_n$. Observe that for $i,j\in\{1,\ldots,k\}$, we have

$$\begin{aligned}\langle w\beta_i,\beta_j\rangle &= \langle a_{1i}\beta_1+\cdots+a_{ki}\beta_k+a_{(k+1)i}\alpha_{k+1}+\cdots+a_{ni}\alpha_n,\beta_j\rangle\\ &= a_{1i}\langle\beta_1,\beta_j\rangle+\cdots+a_{ki}\langle\beta_k,\beta_j\rangle.\end{aligned}$$

Also, for $j\in\{1,\ldots,k\}$, $l\in\{k+1,\ldots,n\}$, we have

$$\begin{aligned}\langle w\alpha_l,\beta_j\rangle &= \langle a_{1l}\beta_1+\cdots+a_{kl}\beta_k+a_{(k+1)l}\alpha_{k+1}+\cdots+a_{ni}\alpha_n,\beta_j\rangle\\ &= a_{1l}\langle\beta_1,\beta_j\rangle+\cdots+a_{kl}\langle\beta_k,\beta_j\rangle.\end{aligned}$$

Let

$$A=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\ \vdots&&\vdots\\ a_{k1}&\cdots&a_{kn}\end{pmatrix},\quad B=\begin{pmatrix}\langle\beta_1,\beta_1\rangle&\cdots&\langle\beta_1,\beta_k\rangle\\ \vdots&&\vdots\\ \langle\beta_k,\beta_1\rangle&\cdots&\langle\beta_k,\beta_k\rangle\end{pmatrix}$$

and

$$D=\begin{pmatrix}\langle w\beta_1,\beta_1\rangle&\cdots&\langle w\beta_k,\beta_1\rangle&\langle w\alpha_{k+1},\beta_1\rangle&\cdots&\langle w\alpha_n,\beta_1\rangle\\ \vdots&&\vdots&\vdots&&\vdots\\ \langle w\beta_1,\beta_k\rangle&\cdots&\langle w\beta_k,\beta_k\rangle&\langle w\alpha_{k+1},\beta_k\rangle&\cdots&\langle w\alpha_n,\beta_k\rangle\end{pmatrix}.$$

Then, noting $B=B^T$, we have $BA=D$. Now, consider wx:

$$wx=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\ \vdots&&\vdots\\ a_{k1}&\cdots&a_{kn}\\ \vdots&&\vdots\\ a_{n1}&\cdots&a_{nn}\end{pmatrix}\begin{pmatrix}c_1\\ \vdots\\ c_k\\ \vdots\\ c_n\end{pmatrix}=\begin{pmatrix}A\\ *\end{pmatrix}\begin{pmatrix}c_1\\ \vdots\\ c_k\\ \vdots\\ c_n\end{pmatrix}=\begin{pmatrix}A\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}\\ *\end{pmatrix},$$

with the entries for $*$ left unspecified as they do not play a role in what follows. That is, if $wx=d_1\beta_1+\cdots+d_k\beta_k+d_{k+1}\alpha_{k+1}+\cdots+d_n\alpha_n$, then

$$\begin{pmatrix}d_1\\ \vdots\\ d_k\end{pmatrix}=A\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}=B^{-1}D\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$$

(noting that B is invertible because $\beta_1, \ldots, \beta_k$ are linearly independent and $\langle \cdot, \cdot \rangle$ is nondegenerate).

We would like to show $B^{-1}D \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \neq \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$, or equivalently, $D \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \neq B \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$.

We have

$$D \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_1\langle w\beta_1, \beta_1\rangle + \cdots + c_k\langle w\beta_k, \beta_1\rangle + c_{k+1}\langle w\alpha_{k+1}, \beta_1\rangle + \cdots + c_n\langle w\alpha_n, \beta_1\rangle \\ \vdots \\ c_1\langle w\beta_1, \beta_k\rangle + \cdots + c_k\langle w\beta_k, \beta_k\rangle + c_{k+1}\langle w\alpha_{k+1}, \beta_k\rangle + \cdots + c_n\langle w\alpha_n, \beta_k\rangle \end{pmatrix}$$

and (using $B = B^T$)

$$B \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} c_1\langle \beta_1, \beta_1\rangle + \cdots + c_k\langle \beta_k, \beta_1\rangle \\ \vdots \\ c_1\langle \beta_1, \beta_k\rangle + \cdots + c_k\langle \beta_k, \beta_k\rangle \end{pmatrix}.$$

Therefore, the ith entry of $B \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} - D \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is equal to

$$\begin{aligned} & c_1\langle \beta_1, \beta_i\rangle + \cdots + c_k\langle \beta_k, \beta_i\rangle \\ & \qquad - [c_1\langle w\beta_1, \beta_i\rangle + \cdots + c_k\langle w\beta_k, \beta_i\rangle + c_{k+1}\langle w\alpha_{k+1}, \beta_i\rangle + \cdots + c_n\langle w\alpha_n, \beta_i\rangle] \\ = \; & c_1\langle \beta_1 - w\beta_1, \beta_i\rangle + \cdots + c_k\langle \beta_k - w\beta_k, \beta_i\rangle - c_{k+1}\langle w\alpha_{k+1}, \beta_i\rangle - \cdots - c_n\langle w\alpha_n, \beta_i\rangle. \end{aligned}$$

Now, $w \in W^{M,A}$ implies that for $j = k+1, \ldots, n$ we have $w\alpha_j > 0$ and hence $\langle w\alpha_j, \beta_i\rangle \geq 0$. Lemma 3.1 tells us that for $j = 1, \ldots, k$, $\langle \beta_j - w\beta_j, \beta_i\rangle \geq 0$. By assumption, $c_1, \ldots, c_k < 0$ and $-c_{k+1}, \ldots, -c_n \leq 0$, so the ith entry is ≤ 0. Now, fix $i \in \{1, \ldots, k\}$ such that $\beta_i - w\beta_i \neq 0$. Since the inner product is symmetric and W-invariant, we have

$$0 < \langle \beta_i - w\beta_i, \beta_i - w\beta_i\rangle = \langle \beta_i, \beta_i\rangle - 2\langle w\beta_i, \beta_i\rangle + \langle \beta_i, \beta_i\rangle = 2\langle \beta_i - w\beta_i, \beta_i\rangle.$$

Therefore, the ith entry is < 0, from which the lemma follows. □

4. Criterion for temperedness

In this section, we give a variation of the Casselman criterion for temperedness (cf. [Cas],[W]). The arguments done later in this paper use exponents in $\mathfrak{a}^*$ (rather than the different $\mathfrak{a}^*_M$ which arise) to facilitate comparison. Thus, in this section, we reformulate the Casselman criterion in terms of exponents in $\mathfrak{a}^*$ (Corollary 4.4) to set up these later arguments. Our starting point is the Cassleman criterion as formulated in Proposition III.2.2. of [W].

Let π be an irreducible admissible representation of G. Let

$$\mathcal{M}_{min}(\pi) = \{L \text{ standard Levi} \,|\, r_{L,G}(\pi) \neq 0 \text{ but } r_{N,G}(\pi) = 0 \text{ for all } N < L\}.$$

Let us define

$$Exp(\pi) = \{\iota(\mu) \,|\, \exp\mu \otimes \rho \leq r_{L,G}(\pi) \text{ for some } \rho \text{ with } \omega_\rho \text{ unitary and } L \in \mathcal{M}_{min}(\pi)\}$$

(where ω_ρ denotes the central character).

We use $\mathcal{E}xp_M(\pi)$ for the exponents of $r_{M,G}(\pi)$ defined by Waldspurger in section I.3 of [W]; $\mathcal{E}xp^{\mathbb{R}}_M(\pi)$ for their real parts.

Lemma 4.1. *Let π be an irreducible representation and $L \in \mathcal{M}_{min}(\pi)$. If $M > L$ a standard Levi factor and $\xi \in \mathcal{E}xp^{\mathbb{R}}_L(\pi)$, then $r^L_M(\xi) \in \mathcal{E}xp^{\mathbb{R}}_M(\pi)$ (where r^L_M denotes restriction from $\mathfrak{a}^*_L$ to $\mathfrak{a}^*_M$), and every $\mu \in \mathcal{E}xp^{\mathbb{R}}_M(\pi)$ has this form (for some $L \in \mathcal{M}_{min}$ and $\xi \in \mathcal{E}xp^{\mathbb{R}}_L(\pi)$).*

Proof. This follows from Proposition 1.9(f) of [B-Z] and (Jacquet) restriction in stages. □

Lemma 4.2. *Let A be the Cartan matrix*

$$A = \begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \dots & \langle \alpha_1, \alpha_n \rangle \\ \vdots & & \vdots \\ \langle \alpha_n, \alpha_1 \rangle & \dots & \langle \alpha_n, \alpha_n \rangle \end{pmatrix}.$$

Then,

$$A^{-1} = \begin{pmatrix} \langle \beta_1, \beta_1 \rangle & \dots & \langle \beta_n, \beta_1 \rangle \\ \vdots & & \vdots \\ \langle \beta_1, \beta_n \rangle & \dots & \langle \beta_n, \beta_n \rangle \end{pmatrix}.$$

In particular, the entries of A^{-1} are nonnegative.

Proof. The characterization of A^{-1} is an immediate consequence of $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$; the non-negativity of its entries is Lemma IV.6.2 of [B-W]. □

Lemma 4.3. *Condition (ii) (for standard parabolics) in Proposition III.2.2 of* [W] *holds if and only if every exponent $\nu \in Exp(\pi)$ satisfies $\nu \in +\bar{\mathfrak{a}}^*$.*

Proof. We check both directions. We remark that both condition (ii) from [W] and the condition here of lying in $+\bar{\mathfrak{a}}^*$ require that the $\mathfrak{z}^*$ component be zero.

$\underline{(\Leftarrow)}$: Let $P = LU$ be a standard parabolic subgroup, with $\Pi_L = \{\alpha_i \mid i \in I_L\}$. If we do not have $L \geq M$ for some $M \in \mathcal{M}_{min}(\pi)$, then $r_{L,G}\pi = 0$ and there is nothing to prove. Thus, we assume $L \geq M$ for some $M \in \mathcal{M}_{min}(\pi)$.

Let $\mu \in \mathfrak{a}_L^* \in \mathcal{E}xp_L^{\mathbb{R}}(\pi)$. By Lemma 4.1, $\mu = r_L^M(\xi)$ for some $\xi \in \mathcal{E}xp_M^{\mathbb{R}}(\pi)$. Then $\nu = \iota_M(\xi) \in \mathfrak{a}^* \in Exp(\pi)$. Note that

$$r_L(\nu) = r_L \circ \iota_M(\xi) = r_L^M \circ r_M \circ \iota_M(\xi) = r_L^M(\xi) = \mu.$$

Write $\nu = z + \sum_{i=1}^n c_i \alpha_i$. Then,

$$\mu = r_L(\nu) = r_L(z) + \sum_{i=1}^n c_i r_L(\alpha_i) = r_L(z) + \sum_{i \notin I_L} c_i \alpha_i^L,$$

where $\alpha_i^L = r_L(\alpha_i)$ (a simple root in $\Pi(P, A_L)$ when $i \notin I_L$). Of course, $\mu \in +\bar{\mathfrak{a}}_L^*$ is then equivalent to

$$c_i \geq 0 \text{ for all } i \notin I_L.$$

On the other hand, the assumption $\nu \in +\bar{\mathfrak{a}}^*$ is equivalent to $c_i \geq 0$ for all i, from which it immediately follows that $\mu \in +\bar{\mathfrak{a}}_L^*$

<u>($\Rightarrow$)</u>: Consider $\nu \in Exp(\pi)$. Then $\nu \in \iota_M(\mathfrak{a}_M^*)$ for some $M \in \mathcal{M}_{min}$. In particular, $\nu \in span_{i \notin I_M}\{\beta_i\}$. Write

$$\nu = \sum_{i=1}^{n} c_i \alpha_i = \sum_{i \notin I_M} d_i \beta_i.$$

Note that our goal is to show $c_i \geq 0$ for all i. If one has $M_{\Pi - \{\alpha_i\}} \geq L$ for some $L \in \mathcal{M}_{min}(\pi)$, then one can use the same basic argument as above to show $c_i \geq 0$. However, this need not hold for all i. In particular, an argument like that above will tell us $c_i \geq 0$ for all $i \notin I_M$; we need to extend this to show $c_i \geq 0$ for all i.

If we let

$$A = \begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \dots & \langle \alpha_1, \alpha_n \rangle \\ \vdots & & \vdots \\ \langle \alpha_n, \alpha_1 \rangle & \dots & \langle \alpha_n, \alpha_n \rangle \end{pmatrix} \text{ and } B = \begin{pmatrix} \langle \beta_1, \beta_1 \rangle & \dots & \langle \beta_1, \beta_n \rangle \\ \vdots & & \vdots \\ \langle \beta_n, \beta_1 \rangle & \dots & \langle \beta_n, \beta_n \rangle \end{pmatrix},$$

then (cf. Lemma 4.2)

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \text{ and } B \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

noting that $d_i = 0$ for $i \in I_M$. For convenience and without loss of generality, suppose the roots are ordered so that $I_M = \{m+1, \dots, n\}$. Let

$$C_1 = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}, \ C_2 = \begin{pmatrix} c_{m+1} \\ \vdots \\ c_n \end{pmatrix} \text{ and } D_1 = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}.$$

We may then write (block matrices)

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \begin{pmatrix} D_1 \\ 0 \end{pmatrix},$$

so that $C_2 = B_{2,1}D_1$. Now,

$$\begin{pmatrix} D_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

implies $D_1 = A_{1,1}C_1 + A_{1,2}C_2$ and therefore

$$C_2 = B_{2,1}D_1 = B_{2,1}A_{1,1}C_1 + B_{2,1}A_{1,2}C_2.$$

Thus, $(I - B_{2,1}A_{1,2})\, C_2 = B_{2,1}A_{1,1}C_1$. It follows from $BA = I$ that $I - B_{2,1}A_{1,2} = B_{2,2}A_{2,2}$. As $A_{2,2}, B_{2,2}$ are invertible (from linear independence considerations, e.g.), this gives

$$C_2 = A_{2,2}^{-1}B_{2,2}^{-1}B_{2,1}A_{1,1}C_1.$$

Again, it follows from $BA = I$ that $B_{2,1}A_{1,1} = -B_{2,2}A_{2,1}$. Therefore,

$$C_2 = -A_{2,2}^{-1}A_{2,1}C_1.$$

Since $A_{2,2}$ is the Cartan matrix for a sub-root system, Lemma 4.2 tells us $A_{2,2}^{-1}$ has nonnegative entries. Also, since $A_{2,1}$ contains no diagonal entries, $-A_{2,1}$ also has nonnegative entries. It now follows from $c_1, \ldots, c_m \geq 0$ that $c_{m+1}, \ldots, c_n \geq 0$, as needed. $\square$

Corollary 4.4. *Let π be an irreducible admissible representation of G. Suppose that the central character of π is unitary. Then π is tempered if and only if every exponent $\nu \in Exp(\pi)$ satisfies $\nu \in +\bar{\mathfrak{a}}^*$.*

5. Multiplicity one in the Jacquet module of a standard module

In this section,we prove the main technical result needed in this paper: if (P, ν, τ) is a set of Langlands data, then $\exp \nu \otimes \tau$ is the only irreducible subquotient $r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ with its central character, and occurs with multiplicity one.

Let θ be an irreducible representation of M. Let us write $|\omega_\theta| = \exp \nu_\theta$, $\nu_\theta \in \mathfrak{a}_M^*$, where ω_θ is the central character of θ. Then, we may (uniquely) write θ as $\exp \nu_\theta \otimes \theta'$ with $\nu_\theta \in \mathfrak{a}_M^*$ and θ' having unitary central character. We call $\iota(\nu_\theta)$ the central exponent for θ (a slight abuse of terminology, as it would be a little more natural to call ν_θ the central exponent). Note that $\exp \nu \otimes \tau$ has central exponent $\iota(\nu)$.

Let *Exp* denote the set of exponents defined in the previous section.

Lemma 5.1. *Let θ be a representation of M and χ a character of M. Then*

$$Exp(\chi \otimes \theta) = \iota(\nu_\chi) + Exp(\theta).$$

Proof. It follows from $r_{L,M}(\chi \otimes \theta) = \chi \otimes r_{L,M}(\theta)$ that

$$\exp \mu \otimes \rho \leq r_{L,M}(\theta) \Leftrightarrow \chi \cdot \exp \mu \otimes \rho \leq r_{L,M}(\chi \otimes \theta).$$

Write $\chi = \exp \nu_\chi \otimes \chi_u$, where χ_u is a unitary character. Then

$$\chi \cdot \exp \mu \otimes \rho = \exp \nu_\chi \cdot \exp \mu \otimes \chi_u \rho = \exp(\nu_\chi + \mu) \otimes \chi_u \rho$$

and the claim follows. □

If $L < M$ is a standard Levi factor, then

$$\mathfrak{a}_L^* = \iota_M^L(\mathfrak{a}_M^*) \oplus (\mathfrak{a}_L^M)^*,$$

where ι_M^L denotes the embedding $\iota_M^L : \mathfrak{a}_M^* \to \mathfrak{a}_L^*$ and $(\mathfrak{a}_L^M)^*$ is the kernel of the restriction $r_M^L : \mathfrak{a}_L^* \to \mathfrak{a}_M^*$ (see section 5 of [A2] for details).

Lemma 5.2. *Let $L < M$. Let θ be an irreducible representation of M and $\mu \in \mathfrak{a}_L^*$ with $\iota(\mu) \in Exp(\theta)$. Write*

$$\mu = \mu_M + \mu_L^M, \quad \mu_M \in \iota_M^L(\mathfrak{a}_M^*), \, \mu_L^M \in (\mathfrak{a}_L^M)^*.$$

If ω_θ is unitary, then $\mu_M = 0$. In general, $\mu_M = \iota_M^L(\nu_\theta)$.

Proof. Suppose ω_θ is unitary. Then $|\omega_\theta| = 1$. According to [Cas], page 45,

$$|\omega_\theta(a)| = \exp \mu(a), \quad a \in A_M.$$

Since $\mu_L^M(a) = 0$, $a \in A_M$, it follows that $\mu_M(a) = 0$, for all $a \in A_M$. Therefore, $\mu_M = 0$.

Now, consider the general case. Write $\theta = \exp \nu_\theta \otimes \theta'$, with θ' having unitary central character. Suppose $\mu \in \mathfrak{a}_L^*$ satisfies $\iota(\mu) \in Exp(\theta)$. Lemma 5.1 tells us $\mu = \iota_M^L(\nu_\theta) + \mu'$, for some $\mu' \in \mathfrak{a}_L^*$ such that $\iota(\mu') \in Exp(\theta')$. Then

$$\mu = \iota_M^L(\nu_\theta) + \mu'_M + (\mu')_L^M.$$

Since $\mu'_M = 0$ and $\nu_\theta \in \mathfrak{a}_M^*$, it follows that $\mu_M = \iota_M^L(\nu_\theta)$. $\square$

Proposition 5.3. *Let $\pi = L(P, \nu, \tau)$. Then $\exp \nu \otimes \tau$ is the unique irreducible subquotient of $r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ having central exponent $\iota(\nu)$, and occurs with multiplicity one.*

Proof. Let $\mathcal{F} = \sum_{i=1}^n \mathbb{R}\alpha_i = \sum_{i=1}^n \mathbb{R}\beta_i$ be as in section 3, so that $\mathfrak{a}^* = \mathfrak{z}^* \oplus \mathcal{F}$. If $\mu \in \mathfrak{a}^*$, we denote by μ^0 the orthogonal projection of μ onto $\mathcal{F}$. Let $I_M = \{i \mid \alpha_i \in \Pi_M\}$. If $\mu \in \mathfrak{a}^*$, then we can write

$$\mu = z + \sum_{i \notin I_M} c_i \beta_i + \sum_{i \in I_M} c_i \alpha_i,$$

where $z \in \mathfrak{z}^*$. In particular, if $\mu \in Exp(\exp \nu \otimes \tau)$, then

$$z + \sum_{i \notin I_M} c_i \beta_i = \iota(\nu), \quad \sum_{i \in I_M} c_i \alpha_i \in Exp(\tau).$$

Since $\nu \in (\mathfrak{a}_M)^*_-$, we have $c_i < 0$ for $i \notin I_M$. Corollary 4.4 implies $c_i \geq 0$ for $i \in I_M$. Therefore, μ^0 satisfies the conditions of Lemma 3.2.

For $\mu = z + \sum_{i \notin I_M} c_i\beta_i + \sum_{i \in I_M} c_i\alpha_i$, let $p_M(\mu) = \sum_{i \notin I_M} c_i\beta_i$. If θ is an irreducible representation of M and $\mu \in Exp(\theta)$, then Lemma 5.2 tells us $\iota(\nu_\theta)^0 = p_M(\mu)$.

It follows from the results of Bernstein-Zelevinsky and Casselman (cf. Lemma 2.12 [B-Z] or section 6 [Cas]) that

$$Exp\left(i_{G,M}(\exp\nu \otimes \tau)\right) \subseteq W^{M,A} \cdot Exp(\exp\nu \otimes \tau).$$

We now combine the above observations. Let $\theta \leq r_{M,G} \circ i_{G,M}(\exp\nu \otimes \tau)$ be irreducible. We have

$$\begin{gathered} Exp(\theta) \subseteq Exp\left(i_{G,M}(\exp\nu \otimes \tau)\right) \subseteq W^{M,A} \cdot Exp(\exp\nu \otimes \tau) \\ \Downarrow \\ CentExp(\theta)^0 \in \left\{p_M\left(W^{M,A} \cdot Exp(\exp\nu \otimes \tau)\right)\right\}, \end{gathered}$$

where $CentExp$ denotes the (M-)central exponent. Thus, to show that $\exp\nu \otimes \tau$ is the unique irreducible subquotient of $r_{M,G} \circ i_{G,M}(\exp\nu \otimes \tau)$ having central exponent $\iota(\nu)$, it suffices to show that $p_M(w\mu) \neq \iota(\nu)^0$ for any $\mu \in Exp(\exp\nu \otimes \tau)$ and $w \in W^{M,A}$ having $w \neq 1$. This follows from Lemma 3.2. $\square$

Corollary 5.4. *Let $\pi = L(P, \nu, \tau)$. Then $\exp\nu \otimes \tau$ is the unique irreducible subquotient of $r_{M,G} \circ i_{G,M}(\exp\nu \otimes \tau)$ having central character $\exp\nu \otimes \omega_\tau$.*

Remark 5.5. Proposition 5.3 and Corollary 5.4 also hold for $O(2n, F)$–this is essentially the same combinatorial statment as for $Sp(2n, F)$ or $SO(2n+1, F)$. In particular, all three have the same Weyl group, the same concrete realization of the Langlands classification (cf. [B-J1] and the appendix to [B-J2]), and the same relevant double-coset representatives for the Weyl group (cf. Lemma 3.6 [Jan3]).

6. The dual Langlands classification

In this section, we give the main result in this paper–the dual Langlands classification (cf. Theorem 6.3).

If θ is an irreducible representation of G with unitary central character, we say that θ is anti-tempered if every exponent $\nu \in Exp(\theta)$ satisfies $\nu \in -\mathfrak{a}_-^*$ (i.e., it satisfies the Casselman criterion with the inequalities reversed–cf. Corollary 4.4). Note that this is equivalent to having $\hat{\theta}$ tempered.

Let $P = MU$ be a standard parabolic subgroup of G. If $w_0 \in W^{M,A}$ is the longest element, then $L = w_0(M)$ is also the Levi factor of a standard parabolic subgroup Q of G. Further, if τ is an irreducible tempered representation of M, then $\theta = w_0\hat{\tau}$ is an irreducible anti-tempered representation of L.

Lemma 6.1. *If $\nu \in (\mathfrak{a}_M)_-^*$, then $\mu = w_0\nu \in (\mathfrak{a}_L)_+^*$.*

Proof. Let $\nu \in (\mathfrak{a}_M)_-^*$ and $\mu = w_0\nu$. If $\gamma \in \Pi(Q, A_L)$, then $\gamma = r_L(\alpha_j)$ for some $\alpha_j \in \Pi - \Pi_L$. Proposition 1.1.4 of [Cas] implies $w_0^{-1}(\alpha_j) < 0$. It follows that $w_0^{-1}(\alpha_j) = \sum_{i=1}^n c_i\alpha_i$, $c_i \leq 0$. Then

$$w_0^{-1}(\gamma) = r_M\left(\sum_{i=1}^n c_i\alpha_i\right) = \sum_{\alpha\in\Pi(P,A_M)} c_\alpha\alpha,$$

where $c_\alpha \leq 0$ and not all c_α are equal to 0. By assumption, $\langle\nu,\alpha\rangle < 0$ for all $\alpha \in \Pi(P, A_M)$. It follows that

$$\langle\mu,\gamma\rangle = \langle w_0^{-1}\mu, w_0^{-1}\gamma\rangle = \langle\nu, \sum_{\alpha\in\Pi(P,A_M)} c_\alpha\alpha\rangle = \sum_{\alpha\in\Pi(P,A_M)} c_\alpha\langle\nu,\alpha\rangle > 0,$$

so $\mu \in (\mathfrak{a}_L)_+^*$. □

Lemma 6.2. *Let $\pi = L(P, \nu, \tau)$. Then $\hat{\pi}$ is the unique irreducible subrepresentation of $i_{G,L}(\exp\mu \otimes \theta)$, with L, μ, θ as above.*

Proof. We have $\exp\nu \otimes \tau \leq r_{M,G}(\pi)$. Corollary 5.4 tells us that $\exp\nu \otimes \tau$ is the unique irreducible subquotient of $r_{M,G}(\pi)$ having central character $\exp\nu \otimes \omega_\tau$. Let Z_M denote the center of M and $Z_L = w_0(Z)$. Combining Lemma 2.1 and Théorème 1.7 of [Aub], we have

$$\exp\mu \otimes \theta = w_0(\exp\nu \otimes \hat{\tau}) = w_0(\widehat{\exp\nu \otimes}\, \tau) \leq r_{L,G}(\hat{\pi}),$$

and this is the unique irreducible subquotient of $r_{L,G} \circ i_{G,L}(\exp\mu \otimes \theta)$ having central character $\exp\mu \otimes \omega_\theta$.

We now need the following standard result ([Cas], [Gus], [W]): If (ρ, V) is an admissible representation of L and ω is a character of Z_L, write

$$V_\omega = \{v \in V \,|\, \text{there is an } n \in \mathbb{N} \text{ such that } [\rho(z) - \omega(z)]^n v = 0 \text{ for all } z \in Z_L\}.$$

Then $V = \oplus_\omega V_\omega$ as a direct sum of L-modules. In particular, let $\rho = r_{L,G}(\hat{\pi})$ and $\lambda = \exp\mu \otimes \omega_\theta$. Then V_λ is just the L-module $\exp\mu \otimes \theta$ (as it is the unique subquotient of $r_{L,G}(\hat{\pi})$ having this central character), so appears as a direct summand in $r_{L,G}(\hat{\pi})$. The lemma now follows from Frobenius reciprocity. $\square$

Theorem 6.3 (The dual Langlands classification)**.** *Let $Q = LU$ be a standard parabolic subgroup of G, $\mu \in (\mathfrak{a}_L)^*_+$, and θ an anti-tempered representation of L. Then the induced representation $i_{G,L}(\exp\nu \otimes \theta)$ has a unique irreducible subrepresentation, which we denote by $DL(Q, \mu, \theta)$. Conversely, if π is an irreducible admissible representation of G, there is a unique triple (Q, μ, θ), with Q a standard parabolic subgroup, $\mu \in (\mathfrak{a}_L)^*_+$ and θ an anti-tempered representation of L, such that $\pi \cong DL(Q, \mu, \theta)$.*

Further, suppose that $\hat{\pi} = L(P, \nu, \tau)$ in the Langlands classification. If $P = MU$ and $w_0 \in W^{M,A}$ is the longest element, we have $L = w_0(M)$, $\mu = w_0\nu$, and $\theta = w_0\hat{\tau}$.

Proof. If (P, ν, τ) is the Langlands data for $\hat{\pi}$, it follows immediately from Lemma 6.2 that (Q, μ, θ) is the dual Langlands data for π. This shows the existence of dual Langlands data. Conversely, if one starts with (Q, μ, θ) is dual Langlands data for

π, Lemma 6.2 implies (P, μ, τ) is Langlands data for $\hat{\pi}$. The uniqueness of dual Langlands data then follows from the uniqueness of Langlands data. The relationship given between the dual Langlands data for π and the Langlands data for $\hat{\pi}$ is immediate from the above discussion. $\square$

Corollary 6.4. *Let $\pi = DL(Q, \mu, \theta)$. Then the multiplicity of π in the induced representation $i_{G,L}(\exp \nu \otimes \theta)$ is one.*

Proof. This follows from the corresponding result for the Langlands classification and the previous theorem. $\square$

Remark 6.5. By Remark 5.5, we know Corollary 5.4 holds for $O(2n, F)$. Further, by [Jan3], we have a duality operator for $O(2n, F)$ with the properties from [Aub] Théorème 1.7. It is then a straightforward matter to check that Lemma 6.2 and Theorem 6.3 hold for $O(2n, F)$ as well.

We close by considering the case of general linear groups. In this case, the dual Langlands classification, suitable interpreted, is the same as the Zelevinsky classification.

We start by reviewing some notation regarding general linear groups, most of which is taken from [Z]. If π_1, π_2 are admissible representations of $GL(k_1, F)$, $GL(k_2, F)$, respectively, we define $\pi_1 \times \pi_2 = i_{G,M}(\pi_1 \otimes \pi_2)$, where $M \cong GL(k_1, F) \times GL(k_2, F)$ is the Levi factor of a standard parabolic subgroup of $G = GL(k_1 + k_2, F)$. Let $\nu = |det|$. Let ρ be an irreducible supercuspidal representation of $GL(m, F)$ and $k \geq 0$ an integer. The set $\Delta = [\rho, \nu^k \rho] = \{\rho, \nu\rho, \ldots, \nu^k \rho\}$ is called a segment. The induced representation $\rho \times \nu\rho \times \cdots \times \nu^k \rho$ has a unique irreducible subrepresentation, which we denote by $\langle \Delta \rangle$, and a unique irreducible quotient, which we denote by $\delta(\Delta)$. For $GL(n, F)$, the Aubert involution coincides with the Zelevinsky involution (cf. Théorème 2.3 [Aub]) and $\widehat{\delta(\Delta)} = \langle \Delta \rangle$. The representation $\delta(\Delta)$ is square integrable if the segment is balanced, i.e., of the form $\Delta = [\nu^{-k}\rho, \nu^k \rho]$, where ρ is unitary and

k is a half-integer. In addition, if τ is a tempered representation of $GL(n, F)$, then $\tau \cong \delta_1 \times \cdots \times \delta_s$, for some square integrable representations $\delta_1, \ldots, \delta_s$; this follows from the irreducibility of induced-from-unitary representations of $GL(n, F)$.

The above implies the following description of the dual Langlands classification for $GL(n, F)$. Suppose $\Delta_1, \Delta_2, \ldots, \Delta_k$ are balanced segments and $\alpha_1 \geq \cdots \geq \alpha_k$ are real numbers. Then the induced representation

$$\nu^{\alpha_1}\langle\Delta_1\rangle \times \nu^{\alpha_2}\langle\Delta_2\rangle \times \cdots \times \nu^{\alpha_k}\langle\Delta_k\rangle$$

has a unique irreducible subrepresentation and any irreducible admissible representation of $GL(n, F)$ can be obtained in this way. If $\alpha_i = \alpha_j$, then $\nu^{\alpha_i}\langle\Delta_i\rangle$ and $\nu^{\alpha_j}\langle\Delta_j\rangle$ may be interchanged; up to such permutations, the inducing data is unique.

Next, we review the Zelevinsky classification (cf. [Z]). We say that the segments Δ_1 and Δ_2 are linked if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$ and $\Delta_1 \cup \Delta_2$ is also a segment. Suppose Δ_1 and Δ_2 are linked and $\Delta_1 = [\rho_1, \nu^{k_1}\rho_1], \Delta_2 = [\rho_2, \nu^{k_2}\rho_2]$. If $\rho_2 = \nu^{\ell}\rho_1$, for some $\ell > 0$, we say that Δ_1 precedes Δ_2.

Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be segments. Suppose for each pair of indices $i < j$, Δ_i does not precede Δ_j. Then the representation $\langle\Delta_1\rangle \times \langle\Delta_2\rangle \times \cdots \times \langle\Delta_k\rangle$ has a unique irreducible subrepresentation which we denote by $\langle\Delta_1, \Delta_2, \ldots, \Delta_k\rangle$. Any irreducible admissible representation of $GL(n, F)$ is isomorphic to some representation of the form $\langle\Delta_1, \Delta_2, \ldots, \Delta_k\rangle$ and the choice of $\Delta_1, \Delta_2, \ldots, \Delta_k$ is unique up to a permutation.

Now, we consider the dual Langlands classification again. Suppose $\Delta_1, \Delta_2, \ldots, \Delta_k$ are balanced segments and $\alpha_1 \geq \cdots \geq \alpha_k$ are real numbers. Let $\Delta_i' = \nu^{\alpha_i}\Delta_i$. It is a straightforward matter to check that $\langle\Delta_1'\rangle, \ldots, \langle\Delta_k'\rangle$ are Zelevinsky data, i.e., $\Delta_1', \ldots, \Delta_k'$ satisfy the do-not-precede condition. Conversely, if we start with Zelevinsky data $\langle\Delta_1'\rangle, \ldots, \langle\Delta_k'\rangle$, we can write $\Delta_i' = \nu^{\alpha_i}\Delta_i$, where Δ_i is balanced and α_i is a real number. Let p be a permutation of $\{1, 2, \ldots, k\}$ such that $\alpha_{p(1)} \geq \cdots \geq \alpha_{p(k)}$. Then $\nu^{\alpha_{p(1)}}\delta(\Delta_{p(1)}), \ldots, \nu^{\alpha_{p(k)}}\delta(\Delta_{p(k)})$ form Langlands data, so $\Delta_{p(1)}', \ldots, \Delta_{p(k)}'$ also satisfy the do-not-precede condition. It follows that

$$\langle \Delta'_1, \Delta'_2, \ldots, \Delta'_k \rangle = \langle \Delta'_{p(1)}, \Delta'_{p(2)}, \ldots, \Delta'_{p(k)} \rangle.$$

The above discussion shows that the dual Langlands classification for $GL(n, F)$, suitably interpreted, is the same as the Zelevinsky classification.

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