

R-GROUPS IN THE NONTEMPERED CASE

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Abstract

In this paper, we study the R-group conjectured by Arthur, associated to the representation parabolically induced from a representation σ . We address the question of which properties of the classical Knapp-Stein R-groups carry over to Arthur's setting, with somewhat surprising results. Some of the basic properties fail: the normalized standard intertwining operators $A(r, \sigma)$, $r \in R$ in general do not form a basis of the commuting algebra and the components of the induced representation are not in a bijective correspondence with the irreducible representations of R . However, the action of standard intertwining operators on the induced space has a natural description in terms of representations of a finite group and the corresponding trace formulation holds.

1. INTRODUCTORY MATERIAL

1.1. Introduction. This paper focuses on questions about R-groups. To set matters up, let F be a p -adic field and G the F -points of a connected quasi-split group defined over F (though we note the results in the third chapter also apply to real groups). Recall that if $P = MN$ is a standard parabolic subgroup of G and σ is a representation of M , we may consider the induced representation $I = \text{Ind}_P^G \sigma$ (normalized parabolic induction). Of particular interest to us in this paper is the case where σ is an irreducible unitary representation, in which case π is also unitary and decomposes as a direct sum.

We start by reviewing the properties of Knapp-Stein R-groups (cf. [Sil2],[Sil3]; [Kn-St] for the real case). Suppose σ is in the discrete series. If W is the Weyl group of G , let $W(\sigma) = \{w \in W \mid w\sigma \cong \sigma\}$ (implicit is $w \cdot M = M$). The R-group is a subgroup of $W(\sigma)$ which determines the intertwining algebra $\text{Hom}_G(I, I)$, among other things. The R-group is defined in terms of Plancherel measures, whence the assumption σ is in the discrete series (see [Sil2],[Sil3] for more details).

For $w \in W(\sigma)$, let $A(\sigma, w)$ denote the normalized standard intertwining operator (cf. [Sha2]). For purposes of this introduction, we assume trivial cocycle, so the normalized standard intertwining operators satisfy $A(\sigma, w_2)A(\sigma, w_1) = A(\sigma, w_2w_1)$ for all $w_1, w_2 \in W(\sigma)$. This is known to hold in a number of important situations (e.g., if σ is generic [Sha2] or $F = \mathbb{R}$ [Kn], [Art2]). In this case, the R-group has the following properties (cf. [Keys] or [Ban2]):

PROPERTIES 1.1. *With notation as above,*

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- (1) *The equivalence classes of components of I are parameterized by the irreducible representations \hat{R} of R . Further, if I_ρ is a component of I corresponding to $\rho \in \hat{R}$, then I_ρ appears with multiplicity $\dim \rho$. That is,*

$$I = \bigoplus (\dim \rho) I_\rho.$$

- (2) *$\text{Hom}_G(I, I) \cong \mathbb{C}[R]$ (with the isomorphism generated by $r \mapsto A(\sigma, r)$ in one direction).*
- (3) *The operators $A(\sigma, r)$ act on and permute the $\dim \rho$ irreducible subspaces of the I_ρ -isotypic component as the representation ρ . This may be expressed in the following manner: for $\pi \cong I_\rho$ an equivalence class of components of I , let*

$$\langle r, \pi \rangle = \text{trace } \rho(r).$$

Then, for $r \in R$, $f \in C_c^\infty(G)$,

$$\text{trace}(A(\sigma, r)I(f)) = \sum_{\pi} \langle r, \pi \rangle \text{trace}(\pi(f)),$$

where the sum runs over the equivalence classes of components of I .

We note that if the cocycle is not trivial, the first statement still applies, along with modified versions of the second and third. In what follows, we also use the above numbering for the corresponding properties when the cocycle is nontrivial.

The theory of the R-group has important applications both locally and globally. Locally, the R-group governs the reducibility of induced discrete series, hence plays a key role in the classification of irreducible tempered representations for real and p -adic groups. Globally, the actions of normalized standard intertwining operators—especially as formulated in (3)—arise (as part of a corresponding global formulation) in the trace formula, hence are of interest in automorphic forms.

Based on global considerations, Arthur conjectured a characterization of the R-group in terms of Arthur parameters (A-parameters), which we refer to as the Arthur R-group (cf. [Art1]). The Arthur R-group is conjectured to exist for more general unitary inducing representations, not just discrete series. Whereas the Knapp-Stein R-group requires the inducing representation to be in the discrete series in order to work with Plancherel measures, A-parameters do not require such a constraint. We note that when the inducing representation is in the discrete series, the Arthur R-group is known to correspond to the Knapp-Stein R-group (with properties (1)-(3) holding) in a number of situations. If $F = \mathbb{R}$, they correspond (cf. [She]). For F p -adic, the Arthur R-group is known to match the Knapp-Stein R-group when G is split and the inducing representation is a character (cf. [Keys]) or for classical groups when the inducing representation is generic (cf. [Ba-Zh]).

A number of results on the Arthur R-group concern certain special nontempered representations (cf. [Jan2],[Ban1],[Ban2],[B-J2]). The basic strategy in these results is to use a duality operator (either the Iwahori-Matsumoto involution [I-M] or the duality of [Aub], [S-S]) to relate $\text{Ind}_P^G \sigma$ to its dual $\widehat{\text{Ind}_P^G \sigma} = \text{Ind}_P^G \hat{\sigma}$ when $\text{Ind}_P^G \sigma$ admits a Knapp-Stein R-group. The properties of the classical Knapp-Stein R-groups are then transferred to the nontempered

representation. In fact, this approach produces more refined results—the R-groups are isomorphic and the action of normalized standard intertwining operators closely related. This is not so surprising given that it is expected that duality admits a nice description in terms of A-parameters.

All the results mentioned above rely on relating Arthur R-groups to Knapp-Stein R-groups. The representations considered are either tempered representations or duals of tempered representations. Arthur’s definition, however, applies to a much wider class of representations, and in the present paper we study the general case. We address the question of which of the properties (1)-(3) carry over to Arthur R-groups in general. Therefore, we do not have some naturally associated Knapp-Stein R-group and we work completely in the context of Arthur parameters.

The results are somewhat surprising. The short answer is that properties (1)-(2) fail to hold in general, while (3) does carry over. We note that it is actually property (3) which is singled out by Arthur.

To show that (1)-(2) do not hold in general, we construct an example where they fail. The representation we consider is

$$\pi = \text{Ind}_P^G(St_{GL(2)} \otimes \text{triv}_{GL(2)}),$$

where $G = SO(9, F)$, $P = MN$ is the standard parabolic subgroup with Levi factor $M \cong GL(2, F) \times GL(2, F)$, $St_{GL(2)}$ is the Steinberg representation of $GL(2, F)$, and $\text{triv}_{GL(2)}$ is the trivial representation of $GL(2, F)$. Using Jacquet module methods (cf. [Tad2],[Jan3],[B-J1],[Muić], etc.), we show π has three components. On the other hand, from [J-S], we know the A-parameter ψ of the inducing representation $\sigma = St_{GL(2)} \otimes \text{triv}_{GL(2)}$, hence may calculate the Arthur R-group $R_{\psi, \sigma}$ for π . We find that

$$R_{\psi, \sigma} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

(not a surprising result in light of [Gol]). Since π has three components but its Arthur R-group has four elements, we see that properties (1)-(2) fail to hold in general.

On the other hand, property (3) does generalize. Needless to say, we do not use properties (1)-(2) in proving (3). Even though properties (1)-(3) have often been discussed together, (3) is somehow more basic—just a property of normalized standard intertwining operators.

We now discuss the results section by section. In the next section, we review some background material on A-parameters, Arthur R-groups, etc. Chapter 2 deals with the example of $\pi = \text{Ind}_P^G(St_{GL(2)} \otimes \text{triv}_{GL(2)})$ mentioned above. Section 2.1 contains a review of the classical groups $SO(2n + 1, F)$ and their Jacquet modules. In 2.2, we use Jacquet module methods to decompose π , showing that it has three components. In 2.3, we calculate its Arthur R-group, showing $R_{\psi, \sigma} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. These combine to show that properties (1)-(2) fail to hold for Arthur R-groups in general. In chapter 3, we show that property (3) (or more precisely, its extension to the nontrivial cocycle case) does hold in general. The proof is based on general results on actions of intertwining operators (Section 3.1) and on Weyl groups for nonconnected reductive groups (Section 3.3). These are then used in 3.4 to show that (3) holds for Arthur R-groups in general.

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his hospitality during her 1-year stay at the Universität Münster, where a part of this work was done.

1.2. Notation and preliminaries. In this section, we introduce notation and recall some results that will be needed in the rest of the paper. Let F be a p -adic field with $\text{char} F = 0$. Let \mathbf{G} be a connected reductive algebraic group defined over F , G its F -points. The group \mathbf{G} is said to be quasi-split if it contains a Borel subgroup which is defined over F ([Sp], Section 3.2). We shall assume that \mathbf{G} is quasi-split over F . Fix a Borel subgroup $B \subset G$ and a maximal split torus $A_0 \subset B$. Let P be a standard parabolic subgroup, i.e., a parabolic subgroup containing B . There exists a unique Levi subgroup in P containing A_0 ; denote it by M . Then M is a connected reductive F -group. We denote by $i_{G,M}$ the functor of normalized parabolic induction ([Be-Ze], Section 2.3). Let A be the split component of M . Denote by $X(M)_F$ and $X(A)_F$ the groups of all F -rational characters of M and A , respectively. Let

$$\mathfrak{a}^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} = X(A)_F \otimes_{\mathbb{Z}} \mathbb{R}$$

and $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$.

We now give a brief discussion of Langlands parameters and Arthur parameters. The reader is referred to [Bor1],[Art1] for a more detailed discussion.

Let \hat{G} denote the dual of G —the complex connected reductive group whose root datum is dual to that of \mathbf{G} . The L-group is then

$${}^L G = \hat{G} \rtimes W_F,$$

where W_F denotes the Weil group of F . Here, the action of W_F on \hat{G} is induced from the action of $\text{Gal}(\bar{F}/F)$ on \mathbf{G} , where \bar{F} is the algebraic closure of F (cf. [Bor1]).

Let

$$\phi : W_F \times SL(2, \mathbb{C}) \longrightarrow {}^L G$$

be a homomorphism. If ϕ commutes with projections to W_F and satisfies the conditions (1)-(5) of [Lan2], we call ϕ a Langlands parameter (L -parameter). We let $\Phi(G)$ denote the set of all equivalence classes of L -parameters of G . If $\Pi(G)$ denotes the set of equivalence classes of irreducible admissible representations of G , the Langlands correspondence predicts that $\Pi(G)$ may be partitioned into disjoint subsets (L -packets) which are in bijective correspondence with $\Phi(G)$. For $\phi \in \Phi(G)$, we let $\Pi_{\phi}(G)$ denote the corresponding L -packet. This bijection is expected to have certain number-theoretic properties characterized in terms of L -functions (cf. [Bor1]).

Let $W'_F = W_F \times SL(2, \mathbb{C})$ denote the Weil-Deligne group. Let

$$\psi : W'_F \times SL(2, \mathbb{C}) \longrightarrow {}^L G$$

be a homomorphism. If $\psi|_{W'_F}$ is an L -parameter and ψ satisfies

- (1) $\psi|_{W'_F}$ is tempered (i.e., the projection of $\psi(W_F)$ to \hat{G} is bounded)
- (2) ψ is algebraic on the second $SL(2, \mathbb{C})$,

we call ψ an Arthur parameter (A -parameter). Let $\Psi(G)$ denote the set of equivalence classes of A -parameters. It is expected that $\Pi(G)$ has subsets (A -packets) which are in bijective correspondence with $\Psi(G)$. Unlike the Langlands correspondence, A -packets need not be

disjoint. If $\psi \in \Psi(G)$, we let $\Pi_\psi(G)$ denote the corresponding A -packet. This correspondence also has certain number-theoretic properties (cf. [Art1]).

We now turn to a discussion of Arthur R -groups. Suppose ψ is an A -parameter of G which factors through ${}^L M = \hat{M} \rtimes W_F \subset {}^L G$,

$$\psi : W'_F \times SL(2, \mathbb{C}) \longrightarrow {}^L M \subset {}^L G.$$

The group ${}^L M$ is the L -group of M . It is a Levi subgroup of ${}^L G$ (see [Bor1], Section 3, for definition of parabolic subgroups and Levi subgroups of ${}^L G$). Then we can regard ψ as an A -parameter of M . Suppose in addition that the image of ψ is not contained in a smaller Levi subgroup (i.e., ψ is an elliptic parameter of M).

Let S_ψ be the centralizer in \hat{G} of the image of ψ and S_ψ^0 its identity component. If T_ψ is a maximal torus of S_ψ^0 , define

$$W_\psi = N_{S_\psi}(T_\psi)/Z_{S_\psi}(T_\psi),$$

$$W_\psi^0 = N_{S_\psi^0}(T_\psi)/Z_{S_\psi^0}(T_\psi).$$

Lemma 2.3 of [Ba-Zh] and the discussion on page 326 of [Ba-Zh] imply that W_ψ can be identified with a subgroup of $W(G, A)$.

Let σ be an irreducible unitary representation of M . Assume σ belongs to the A -packet $\Pi_\psi(M)$. If $W(\sigma) = \{w \in W(G, A) \mid w\sigma \cong \sigma\}$, we let

$$W_{\psi, \sigma} = W_\psi \cap W(\sigma)$$

$$W_{\psi, \sigma}^0 = W_\psi^0 \cap W(\sigma)$$

and take

$$R_{\psi, \sigma} = W_{\psi, \sigma}/W_{\psi, \sigma}^0$$

as the Arthur R -group.

2. AN EXAMPLE

In this section, we give an example which shows that not all properties of classical R -groups carry over to the nontempered setting. In particular, we consider the representation $St_{GL(2)} \times \text{triv}_{GL(2)} \rtimes 1$ (cf. section 2.1 for notation) of $SO(9, F)$. Using Jacquet module methods, we show that this representation has 3 components (cf. Theorem 2.5). A calculation of the Arthur R -group shows that $R_{\psi, \sigma} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (cf. section 2.3). In particular, $|R_{\psi, \sigma}|$ does not give the number of components, so Properties (1)-(2) from the introduction fail.

2.1. Classical groups. In this section, we review background on special odd-orthogonal groups which will be needed in the rest of this chapter.

We define \times on general linear groups as in [Be-Ze]: if ρ_1, \dots, ρ_k are representations of $GL(n_1, F), \dots, GL(n_k, F)$, let $\rho_1 \times \dots \times \rho_k$ denote the representation of $GL(n_1 + \dots + n_k, F)$ obtained by inducing $\rho_1 \otimes \dots \otimes \rho_k$ from the standard parabolic subgroup of $GL(n_1 + \dots + n_k, F)$ with Levi factor $GL(n_1, F) \times \dots \times GL(n_k, F)$.

In much of 2.2, we work in the Grothendieck group setting. That is, we work with the semisimplified representation. So, for any representation π and irreducible representation ρ ,

to be applied to representations M_β when $M_\beta > M_\alpha$ (cf. section 2.1, [Be-Ze]). We will occasionally use similar notation for representations of $GL(n, F)$. If $\alpha = (n_1, \dots, n_k)$ is a partition of $m \leq n$, $GL(n, F)$ has a standard parabolic subgroup with Levi factor $L_\alpha \cong GL(n_1, F) \times \dots \times GL(n_k, F) \times GL(n - m, F)$ (L_α consists of block-diagonal matrices; the corresponding parabolic subgroup of block upper triangular matrices). If π is a representation of $GL(n, F)$, we let $r_\alpha(\pi)$ denote the Jacquet module of π with respect to L_α .

We now give the Langlands classification for $GL(n, F)$ and $SO(2n + 1, F)$ (cf. [B-W],[Sil1]; for real groups, see [Lan1]). As in [Zel], let $\nu = |\det|$ on $GL(n, F)$ (with the value of n clear from context). Suppose that δ is an irreducible essentially square integrable representation of $GL(n, F)$. Then, there is an $\varepsilon(\delta) \in \mathbb{R}$ such that $\nu^{-\varepsilon(\delta)}\delta$ is unitarizable. For $GL(n, F)$, let $\delta_1, \dots, \delta_k$ be irreducible essentially square integrable representations satisfying $\varepsilon(\delta_1) \leq \dots \leq \varepsilon(\delta_k)$. Then, $\delta_1 \times \dots \times \delta_k$ has a unique irreducible subrepresentation which we denote by $\mathcal{L}(\delta_1, \dots, \delta_k; \tau)$. For $SO(2n + 1, F)$, let $\delta_1, \dots, \delta_k$ be irreducible essentially square integrable representations satisfying $\varepsilon(\delta_1) \leq \dots \leq \varepsilon(\delta_k) < 0$ and τ a tempered representation of $SO(2(n - m) + 1, F)$. Then, $\delta_1 \times \dots \times \delta_k \rtimes \tau$ has a unique irreducible subrepresentation which we denote by $L(\delta_1, \dots, \delta_k; \tau)$. Note that we use Langlands classification in the subrepresentation setting rather than the quotient setting for the following reason: in the subrepresentation setting, $\delta_1 \otimes \dots \otimes \delta_k \otimes \tau$ will lie in the appropriate Jacquet module of $L(\delta_1, \dots, \delta_k; \tau)$ (by Frobenius reciprocity).

We now recall some structure theory related to Jacquet modules.

DEFINITION 2.1. (1) *If τ is a representation of $GL(n, F)$, set*

$$m^* \tau = \sum_{i=0}^n r_{(i)} \tau$$

(2) *If π is a representation of $SO(2n + 1, F)$, set*

$$\mu^* \pi = \sum_{i=0}^n s_{(i)} \pi.$$

If τ_1 and τ_2 are representations of $GL(n_1, F)$, $GL(n_2, F)$, respectively, let $s(\tau_1 \otimes \tau_2) = \tau_2 \otimes \tau_1$ and $m(\tau_1 \otimes \tau_2) = \tau_1 \times \tau_2$. If τ is a representation of $GL(n, F)$ and ϑ is a representation of $SO(2m + 1, F)$, define \rtimes on $(R \otimes R) \otimes (R \otimes R[S])$ by $(\tau_1 \otimes \tau_2) \rtimes (\tau \otimes \vartheta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \vartheta)$. Set $M_S^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*$ (\sim denotes contragredient).

THEOREM 2.2 (Tadić). *If τ is a representation of $GL_n(F)$ and ϑ a representation of S_m , then*

$$\mu^*(\tau \rtimes \vartheta) = M_S^*(\tau) \rtimes \mu^*(\vartheta).$$

Proof. See [Tad1]. \square

We mention the counterpart for general linear groups: if we define \times on $R \otimes R$ by $(\tau_1 \otimes \tau_2) \times (\tau'_1 \otimes \tau'_2) = (\tau_1 \times \tau'_1) \otimes (\tau_2 \times \tau'_2)$, then $m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \times m^*(\pi_2)$. See section 1.7 of [Zel].

2.2. Decomposition of $St_{GL(2)} \times triv_{GL(2)} \rtimes 1$. We analyze the induced representation $St_{GL(2)} \times triv_{GL(2)} \rtimes 1$ using Jacquet module methods (cf. [Tad2],[Jan3],[B-J1],[Muić], etc., for similar arguments). Before getting into the analysis, we first introduce a couple of representations which will occur.

By Theorem 4.5 [Jan1] or Lemma 4.2 [Jan3], the degenerate principal series $triv_{GL(2)} \rtimes triv_{SO(3)}$ is irreducible. Therefore, by duality (cf. [Aub],[S-S]) or the Iwahori-Matusmoto involution, we see that

$$\mathcal{S} = St_{GL(2)} \rtimes St_{SO(3)}$$

is an irreducible tempered representation. Similarly, the degenerate principal series $triv_{GL(2)} \rtimes 1$ decomposes as $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ (cf. Theorem 4.1 [Jan1] or Proposition 3.6 [Jan3]). Taking duals,

$$St_{GL(2)} \rtimes 1 = \mathcal{T}_1 + \mathcal{T}_2,$$

where $s_{(1)}\mathcal{T}_1 = \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}})$ and $s_{(1)}\mathcal{T}_2 = \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes St_{SO(3)}$. We summarize:

- LEMMA 2.3.** (1) $\nu^{-\frac{1}{2}} \rtimes triv_{SO(3)} = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + \mathcal{T}_1$
(2) $triv_{GL(2)} \rtimes 1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + L(\nu^{-\frac{1}{2}}; St_{SO(3)})$
(3) $\nu^{-\frac{1}{2}} \rtimes St_{SO(3)} = L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \mathcal{T}_2$
(4) $St_{GL(2)} \rtimes 1 = \mathcal{T}_1 + \mathcal{T}_2$

The Jacquet modules for the irreducible representations appearing above are given in the table at the end of this section.

LEMMA 2.4. *We have the following:*

- (1) $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)})$
(2) $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$
(3) $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}) = L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$
(4) $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_1)$
(5) $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) + \mathcal{S}$.

The Jacquet modules for the irreducible representations appearing above are given in the table at the end of this section.

Proof. For (1), observe that since $s_{(1)}L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = \nu^{-\frac{1}{2}} \otimes St_{SO(3)}$, by Theorem 2.2,

$$\begin{aligned} s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) &= \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) \\ &\quad + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes St_{SO(3)} \\ &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) \end{aligned}$$

(cf. Lemma 2.3). Now,

$$s_{(1,1,1)}\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$$

$$s_{(1,1,1)}\mathcal{T}_2 = \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} + 2\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$$

$$s_{(1,1,1)}\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}.$$

Let π_1 be an irreducible subquotient of $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ such that $s_{(1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$. Observe that the only irreducible representation of $GL(3, F)$ having $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ in its Jacquet module is $\mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}) = \nu^{-\frac{1}{2}} \times \mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}})$. Since $r_{(1,1)}\mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}) = 2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$, we have $s_{(1,1,1)}\pi_1 \geq 2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$. Therefore, $s_{(1)}\pi_1 \geq 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$. However, this then tells us $s_{(1,1,1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ (consider $s_{(1,1,1)}\nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$). Now, the only irreducible representation of $GL(3, F)$ having $\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ in its Jacquet module is $\mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}) = \mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}) \times \nu^{\frac{1}{2}}$. Since $r_{(1,1)}\mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}) = 2\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$, we have $s_{(1,1,1)}\pi_1 \geq \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$. Therefore, $s_{(1)}\pi_1$ also contains $\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$. As $s_{(1)}\pi_1$ accounts for all of $s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$, we see that π_1 is the only component, i.e., $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ is irreducible. Further, since

$$\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes St_{SO(3)}$$

has $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)})$ as unique irreducible subrepresentation (by the Langlands classification), we must have

$$\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)}),$$

as claimed.

For (2), since $s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) = 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes St_{SO(3)}$, Theorem 2.2 tells us

$$\begin{aligned} s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) &= \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}) \\ &\quad + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes St_{SO(3)} \\ &= 3\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + 2\nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 \\ &\quad + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}), \end{aligned}$$

by Lemma 2.3. Let π_1 be an irreducible subquotient of $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ such that $s_{(1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$. Then, $s_{(1,1,1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$. By Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi_1, \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes 1) \cong \mathrm{Hom}_A(s_{(1,1,1)}\pi_1, \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}),$$

which is nonzero by a central character argument (cf. Lemma 8.2 [Gus]). Therefore,

$$\pi_1 \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes 1.$$

Since $\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes 1$ has unique irreducible subrepresentation $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ (by the Langlands classification), we see that $\pi_1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$. Now, $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) = \mathrm{triv}_{GL(2)} \rtimes \mathrm{triv}_{SO(3)}$ and has

$$s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) = 3\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + 2\nu^{-\frac{1}{2}} \otimes \mathcal{T}_1$$

(which follows from Theorem 2.2 and Lemma 2.3). This leaves $\nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$ and $\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ unaccounted for. An $s_{(1,1,1)}$ argument like that used for part (1) shows that if π_2 is the irreducible subquotient with $s_{(1)}\pi_2 \geq \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$, then $s_{(1)}\pi_2$ must also contain $\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$.

Further, by central character considerations like those used above,

$$0 \neq \text{Hom}_M(r_{M,G}\pi_2, \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2) \cong \text{Hom}_G(\pi_2, \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2),$$

so $\pi_2 \hookrightarrow \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2$. Thus, the Langlands classification tells us $\pi_2 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$, as claimed.

We now turn to (3). By Theorem 2.2, we have

$$\begin{aligned} s_{(1)}St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}) &= 2\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes St_{GL(2)} \rtimes 1 \\ &= 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 \end{aligned}$$

by Lemma 2.3. Observe that $L(\nu^{-\frac{1}{2}}; \mathcal{T}_1)$ is the unique irreducible subrepresentation of $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$ (by the Langlands classification) and $s_{(1)}L(\nu^{-\frac{1}{2}}; \mathcal{T}_1)$ does not contain $\nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$ (since $s_{(1)}\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$ does not—an easy calculation, or cf. Lemma 3.4 [Jan4]). The corresponding statement holds for $L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$. Let π_1 be the component of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}})$ such that $s_{(1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1$. Then, $s_{(1)}\pi_1 \not\geq \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$. Therefore, by the same central character considerations used in (2), we have $\pi_1 \hookrightarrow \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$, hence $\pi_1 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_1)$. Similarly, we see $\pi_2 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$ is also a component of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}})$. Further, since $s_{(1)}\pi_2 = \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ (cf. (2) above), we have

$$s_{(1)}\pi_1 \leq \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes \mathcal{T}_1.$$

Again, an $s_{(1,1,1)}$ argument like that used in part (1) tells us

$$s_{(1)}\pi_1 = \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes \mathcal{T}_1.$$

Thus, π_1 and π_2 are the only components, so

$$St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}) = L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + L(\nu^{-\frac{1}{2}}; \mathcal{T}_2),$$

as claimed.

For (4) and (5), note that the duals (in the sense of [Aub],[S-S]) to $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$ and $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2$ are $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ and $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$, resp., which have been analyzed in (1) and (2). As $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ is irreducible, so is $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$. Therefore,

$$\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_1).$$

Similarly, as $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ has two irreducible subquotients, so does $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2$. One of them must be $L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$. Since $triv_{GL(2)} \rtimes triv_{SO(3)}$ is the other irreducible subquotient of $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$, its dual—i.e., $St_{GL(2)} \rtimes St_{SO(3)} = \mathcal{S}$ —is the other irreducible subquotient of $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2$. This finishes the proof of the lemma. \square

THEOREM 2.5.

$$St_{GL(2)} \times triv_{GL(2)} \rtimes 1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1) + L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2) + L(\nu^{-\frac{1}{2}}; \mathcal{S})$$

The Jacquet modules of the components are given in the table at the end of this section.

Proof. By Lemma 2.3,

$$St_{GL(2)} \times \text{triv}_{GL(2)} \rtimes 1 = St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}).$$

Thus, it suffices to analyze these two induced representations.

Let us start with $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$. By Theorem 2.2,

$$\begin{aligned} s_{(1)}St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) &= 2\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes St_{GL(2)} \rtimes St_{SO(3)} \\ &= 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes \mathcal{S} \end{aligned}$$

by Lemma 2.4. Now, let π_1 be a component such that $s_{(1)}\pi_1 \geq \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)})$. Then,

$$\begin{aligned} s_{(1,1,1)}\pi_1 &\geq \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)} \\ &\quad \Downarrow \\ s_{(3)}\pi_1 &\geq \mathcal{L}(\nu^{-\frac{1}{2}}, St_{GL(2)}) \otimes St_{SO(3)}, \end{aligned}$$

since $\mathcal{L}(\nu^{-\frac{1}{2}}, St_{GL(2)}) = \nu^{-\frac{1}{2}} \times St_{GL(2)}$ is the only irreducible representation of $GL(3, F)$ containing $\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$ in its Jacquet module. Now, the Jacquet module of $\mathcal{L}(\nu^{-\frac{1}{2}}, St_{GL(2)})$ also contains $\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$. Therefore,

$$\begin{aligned} s_{(1,1,1)}\pi_1 &\geq \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)} \\ &\quad \Downarrow \\ s_{(1)}\pi_1 &\geq \nu^{-\frac{1}{2}} \otimes \mathcal{S}. \end{aligned}$$

Thus, there is only the single component π_1 , i.e., $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ is irreducible. By the now usual central character considerations, $\pi_1 \hookrightarrow \nu^{-\frac{1}{2}} \rtimes \mathcal{S}$. Therefore, by the Langlands classification,

$$St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = \pi_1 = L(\nu^{-\frac{1}{2}}; \mathcal{S}).$$

We now turn to $\pi = St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$. By Theorem 2.2,

$$\begin{aligned} s_{(1)}St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) &= 2\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{-\frac{1}{2}} \otimes St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}) \\ &\quad + \nu^{-\frac{1}{2}} \otimes St_{GL(2)} \rtimes St_{SO(3)} \\ &= 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) \\ &\quad + 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) + \nu^{-\frac{1}{2}} \otimes \mathcal{S} \end{aligned}$$

by Lemma 2.4. We observe that the only terms of the form $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \tau$ (τ irreducible) in $s_{(1,1)}\pi$ are $2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1$ and $2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$. Therefore, by central character considerations,

$$0 \neq \text{Hom}_M(r_{M,G}\pi, \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \mathcal{T}_i) \cong \text{Hom}_G(\pi, \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_i)$$

for some i . That is, there is component π_i of π such that $\pi_i \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_i$; by the Langlands classification, $\pi_i = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_i)$. This shows $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_i)$ is a component of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ for some i . We next show that this actually holds for both $i = 1, 2$.

Observe that

$$\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_i) \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \mathcal{T}_i.$$

As $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_i)$ is the unique irreducible subrepresentation of $\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \mathcal{T}_i$, we must have

$$L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_i) \hookrightarrow \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_i).$$

Therefore, $s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_i) \leq s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_i)$. Now, by Theorem 2.2 and Lemma 2.4,

$$\begin{aligned} s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) &= \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1 \\ &\quad + 2\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1 + \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) \\ &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + 3\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + \nu^{\frac{1}{2}}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) \\ &\quad + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) \end{aligned}$$

and

$$\begin{aligned} s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) &= \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2 \\ &\quad + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) \\ &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) + 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) + \nu^{-\frac{1}{2}} \otimes \mathcal{S} \\ &\quad + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}). \end{aligned}$$

Suppose, e.g., $\pi_1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1)$. The above calculations then show $\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) \not\leq s_{(1)}\pi_1$. So, we let π_2 be a component such that $s_{(1)}\pi_2 \geq \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$. The same central character/Frobenius reciprocity argument used above then tells us $\pi_2 \hookrightarrow \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$, hence $\pi_2 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2)$. We note that the same considerations would apply if we started with $\pi_1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2)$. Thus, both $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1)$ and $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2)$ are components of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$, as claimed.

It remains to show that $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1)$ and $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2)$ are the only components. This may be done using the same sort of $s_{(1,1,1)}$ considerations applied in showing the irreducibility of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$. \square

Jacquet modules:

$$\begin{aligned} s_{(1)}L(\nu^{-\frac{1}{2}}; \mathcal{S}) &= \nu^{-\frac{1}{2}} \otimes \mathcal{S} + 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)}) \\ s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1) &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) \\ s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2) &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) + \nu^{-\frac{1}{2}} \otimes \mathcal{S} + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) \\ s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) &= 3\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + 2\nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 \\ s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)}) &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) \\ s_{(1)}L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) &= \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes \mathcal{T}_1 \\ s_{(1)}L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) &= \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) \\ s_{(1)}\mathcal{S} &= 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{\frac{1}{2}} \otimes \mathcal{T}_1 + 3\nu^{\frac{1}{2}} \otimes \mathcal{T}_2 \\ s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes St_{SO(3)} \\ s_{(1)}L(\nu^{-\frac{1}{2}}; St_{SO(3)}) &= \nu^{-\frac{1}{2}} \otimes St_{SO(3)} \\ s_{(1)}\mathcal{T}_1 &= \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) \\ s_{(1)}\mathcal{T}_2 &= \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes St_{SO(3)} \end{aligned}$$

REMARK 2.6. *The calculations done in this section can be generalized to (ρ, σ) reducibility at $1/2$ (cf. [Tad2] for more details).*

2.3. Calculation of the Arthur R-group. We now calculate the Arthur R -group for $St_{GL(2)} \times triv_{GL(2)} \rtimes 1$. Now, $\sigma = St_{GL(2)} \otimes triv_{GL(2)}$ is a representation of the standard parabolic subgroup with Levi factor $M \cong GL(2, F) \times GL(2, F) \subset SO(9, F)$. Then, $\hat{M} \cong GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \subset Sp(8, \mathbb{C})$. Thus, the associated A-parameter

$$\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \longrightarrow \hat{M} \subset Sp(8, \mathbb{C})$$

may be determined from [J-S]. It has the form

$$\psi = (1 \otimes S_2 \otimes S_1) \oplus (1 \otimes S_1 \otimes S_2) \oplus (1 \otimes S_1 \otimes S_2) \oplus (1 \otimes S_2 \otimes S_1),$$

where 1 denotes the trivial representation of W_F and S_n the standard irreducible n -dimensional representation of $SL(2, \mathbb{C})$. In particular, S_1 is the trivial representation and S_2 may be chosen to have $S_2(x) = x$ for $x \in SL(2, \mathbb{C})$. Thus,

$$image \psi = \left\{ \left(\begin{array}{cccc} X_1 & & & \\ & X_2 & & \\ & & \tau X_2^{-1} & \\ & & & \tau X_1^{-1} \end{array} \right) \mid X_1, X_2 \in SL(2, \mathbb{C}) \right\},$$

where τ denotes transpose with respect to the antidiagonal.

Given the form of $image \psi$, its centralizer has the form

$$S_\psi = \begin{pmatrix} A_1 & & & B_1 \\ & A_2 & B_2 & \\ & C_2 & D_2 & \\ C_1 & & & D_1 \end{pmatrix},$$

with A_i, B_i, C_i, D_i 2×2 matrices. In particular, $W_\psi, W_\psi^0 \subset W(\sigma)$. Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that $EX = \tau X^{-1}E$ for $X \in SL(2, \mathbb{C})$ (i.e., E gives the equivalence of representations $S_2 \cong \tau S_2^{-1}$, in the obvious notation). Now,

$$\begin{aligned} & \begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \tau X_2^{-1} & \\ & & & \tau X_1^{-1} \end{pmatrix} \begin{pmatrix} A_1 & & & B_1 \\ & A_2 & B_2 & \\ & C_2 & D_2 & \\ C_1 & & & D_1 \end{pmatrix} \\ &= \begin{pmatrix} A_1 & & & B_1 \\ & A_2 & B_2 & \\ & C_2 & D_2 & \\ C_1 & & & D_1 \end{pmatrix} \begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \tau X_2^{-1} & \\ & & & \tau X_1^{-1} \end{pmatrix} \end{aligned}$$

immediately gives

$$\begin{aligned} X_i A_i &= A_i X_i & \tau X_i^{-1} C_i &= C_i X_i \\ X_i B_i &= B_i \tau X_i^{-1} & \tau X_i^{-1} D_i &= D_i \tau X_i^{-1}. \end{aligned}$$

Therefore, by Schur's lemma,

$$\begin{aligned} A_i &= \lambda_{A_i} I & C_i &= \lambda_{C_i} E \\ B_i &= \lambda_{B_i} E & D_i &= \lambda_{D_i} I \end{aligned}$$

for scalars $\lambda_{A_i}, \lambda_{B_i}, \lambda_{C_i}, \lambda_{D_i}$ (with I the 2×2 identity matrix).

Now, to have

$$\begin{pmatrix} \lambda_{A_1} I & & & \lambda_{B_1} E \\ & \lambda_{A_2} I & \lambda_{B_2} E & \\ & \lambda_{C_2} E & \lambda_{D_2} I & \\ \lambda_{C_1} E & & & \lambda_{D_1} I \end{pmatrix} \in Sp(8, \mathbb{C}),$$

we must have $\lambda_{A_i} \lambda_{C_i} = \lambda_{B_i} \lambda_{D_i} = 0$ and $\lambda_{A_i} \lambda_{D_i} + \lambda_{B_i} \lambda_{C_i} = 1$. Thus,

$$\begin{aligned} S_\psi &= \left\{ \begin{pmatrix} \lambda_{A_1} I & & & \\ & \lambda_{A_2} I & & \\ & & \lambda_{A_2}^{-1} I & \\ & & & \lambda_{A_1}^{-1} I \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \lambda_{A_1} I & & & \\ & & \lambda_{B_2} E & \\ & & \lambda_{B_2}^{-1} E & \\ & & & \lambda_{A_1}^{-1} I \end{pmatrix} \right\} \\ &\cup \left\{ \begin{pmatrix} & & & \lambda_{A_1}^{-1} I \\ & \lambda_{A_2} I & & \lambda_{B_1} E \\ & & \lambda_{A_2}^{-1} I & \\ \lambda_{B_1}^{-1} E & & & \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} & & & \lambda_{A_1}^{-1} I \\ & \lambda_{A_2} I & & \lambda_{B_1} E \\ & & \lambda_{A_2}^{-1} I & \\ \lambda_{B_1}^{-1} E & & & \lambda_{B_2} E \end{pmatrix} \right\}. \end{aligned}$$

Therefore, $W_{\psi, \sigma} = W_\psi \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $W_{\psi, \sigma}^0 = W_\psi^0 = 1$. Thus, the Arthur R-group is

$$R_{\psi, \sigma} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

as claimed.

3. ACTION OF INTERTWINING OPERATORS

In this section, we show that (3) of Properties 1.1 holds for intertwining operators. More precisely, we show the appropriate generalization to accommodate nontrivial cocycle holds. Our approach is based on arguments used for Knapp-Stein R-groups (cf. [Keys], [Ke-Sh], [Ban2]) and the adaptations to nontrivial cocycle from [Art3]. We note that the results in this section also apply to real groups (noting that the accommodations for nontrivial cocycle are not needed for real groups).

G, M are as in Section 1.2

3.1. Actions on intertwining algebras. Let σ be an irreducible unitary representation of M and consider the induced representation $I = i_{G,M}(\sigma)$. Let V denote the representation space of I . Let R be a finite group, $\mathbb{C}[R]$ its group algebra. Suppose

$$r \mapsto A(r)$$

extended linearly to $\mathbb{C}[R]$ gives a homomorphism of $\mathbb{C}[R]$ into the intertwining algebra $C(\sigma) = \text{Hom}(I, I)$. In particular, this gives rise to a representation of R on V which commutes with the action of G . It follows $\tilde{I} : R \times G \rightarrow \text{Aut}(V)$ given by

$$\tilde{I}(r, g) = A(r)I(g)$$

is a representation of $R \times G$ on V . We have

$$(1) \quad \tilde{I} \cong \bigoplus_{\rho \otimes \pi} m_{\rho \otimes \pi} \rho \otimes \pi,$$

where ρ runs over the set of equivalence classes of irreducible representations of R and π runs over the set of equivalence classes of irreducible components of I . The integer $m_{\rho \otimes \pi} \geq 0$ is the multiplicity of $\rho \otimes \pi$ in \tilde{I} . Let

$$(2) \quad V = \bigoplus_{\rho \otimes \pi} \bigoplus_{i=1}^{m_{\rho \otimes \pi}} V_{\rho \otimes \pi}(i) \cong \bigoplus_{\rho \otimes \pi} \bigoplus_{i=1}^{m_{\rho \otimes \pi}} U_{\rho \otimes \pi}^i \otimes V_{\rho \otimes \pi}^i$$

be a decomposition of V into \tilde{I} -irreducible subspaces corresponding to (1), with $U_{\rho \otimes \pi}^i$ (resp., $V_{\rho \otimes \pi}^i$) an irreducible R -invariant (resp., G -invariant) subspace of $V_{\rho \otimes \pi}(i)$. For an irreducible component π of I , define

$$(3) \quad \rho_\pi = \bigoplus_{\rho} m_{\rho \otimes \pi} \rho.$$

This representation acts on the space $U_\pi = \bigoplus_{\rho} \bigoplus_{i=1}^{m_{\rho \otimes \pi}} U_{\rho \otimes \pi}^i \subset V$. The spaces $V_{\rho \otimes \pi}^i$, for all ρ and $i = 1, \dots, m_{\rho \otimes \pi}$, are mutually equivalent and we can identify each of them with a space denoted by V_π ; we write $I_{\rho,i} : V_\pi \rightarrow V_{\rho \otimes \pi}^i$ for the maps. From (2),

$$V \cong \bigoplus_{\rho \otimes \pi} \bigoplus_{i=1}^{m_{\rho \otimes \pi}} U_{\rho \otimes \pi}^i \otimes V_\pi = \bigoplus_{\pi} \left(\bigoplus_{\rho} \bigoplus_{i=1}^{m_{\rho \otimes \pi}} U_{\rho \otimes \pi}^i \right) \otimes V_\pi = \bigoplus_{\pi} U_\pi \otimes V_\pi.$$

We may describe the isomorphism explicitly: for $u \in U_\pi$, write $u = \sum_{\rho} \sum_i u_{\rho,i}$ with $u_{\rho,i} \in U_{\rho \otimes \pi}^i$. For $v \in V_\pi$, we have $u \otimes v \in U_\pi \otimes V_\pi$ corresponds to

$$\mathcal{I}(u \otimes v) = \sum_{\rho} \sum_i u_{\rho,i} \otimes I_{\rho,i}^{-1}(v) \in V,$$

which extends to give the isomorphism. It follows that

$$(4) \quad \tilde{I} \cong \bigoplus_{\pi} \rho_\pi \otimes \pi, \quad I \cong \bigoplus_{\pi} (\dim \rho_\pi) \pi.$$

For a given π , the space $U_\pi \otimes V_\pi$ is the π -isotypic subspace of V and therefore is canonically defined (although U_π and V_π are not). The representation ρ_π is also canonical. If we want to fix a decomposition of $U_\pi \otimes V_\pi$ into G -irreducible subspaces, we fix an orthonormal basis

$\{u_1, \dots, u_k\}$ of U_π . Then $U_\pi \otimes V_\pi = (u_1 \otimes V_\pi) \oplus \dots \oplus (u_k \otimes V_\pi)$. The action of $A(r)$ on $u \otimes v \in u \otimes V_\pi$ is precisely $\rho_\pi(r)u \otimes v$.

For $f \in C_c^\infty(G)$, define

$$I(f) = \int_G f(g)I(g)dg, \quad \pi(f) = \int_G f(g)\pi(g)dg.$$

For $v \in V$, write $\mathcal{I}^{-1}v = \sum_\pi \sum_i u_\pi^i \otimes v_\pi^i$, in accordance with the decomposition (4) of \tilde{I} ($u_\pi^i \otimes v_\pi^i \in U_\pi \otimes V_\pi$). Then, for $r \in R$ and $f \in C_c^\infty(G)$, we have

$$\begin{aligned} A(r)I(f)v &= A(r) \int_G f(g)I(g)v dg \\ &= \int_G f(g)A(r)I(g)v dg \\ &= \int_G f(g)\tilde{I}(r, g)v dg \\ &= \int_G f(g) \sum_\pi \sum_i \mathcal{I}(\rho_\pi(r)u_\pi^i \otimes \pi(g)v_\pi^i) dg \\ &= \sum_\pi \sum_i \mathcal{I}\left(\rho_\pi(r)u_\pi^i \otimes \int_G f(g)\pi(g)v_\pi^i\right) dg \\ &= \sum_\pi \sum_i \mathcal{I}(\rho_\pi(r)u_\pi^i \otimes \pi(f)v_\pi^i) \\ &= \left(\bigoplus_\pi \rho_\pi(r) \otimes \pi(f)\right)v. \end{aligned}$$

Notice that all the integrals above are essentially finite sums. It follows that

$$A(r)I(f) = \bigoplus_\pi \rho_\pi(r) \otimes \pi(f).$$

In particular,

$$\text{trace}(A(r)I(f)) = \sum_\pi \text{trace}(\rho_\pi(r) \otimes \pi(f)) = \sum_\pi \text{trace} \rho_\pi(r) \text{trace} \pi(f).$$

We have proved the following:

LEMMA 3.1. *Let σ be an irreducible admissible unitary representation of M and $I = i_{G,M}(\sigma)$. Let R be a finite group. Suppose $r \mapsto A(r)$ is a homomorphism of R into a multiplicative subgroup of $C(\sigma) = \text{Hom}(I, I)$. To each component π of I we can attach in a canonical way a representation ρ_π of R . Then*

$$I = \bigoplus_\pi (\dim \rho_\pi)\pi,$$

where π runs over equivalence classes of irreducible subrepresentations of I . If we define

$$\langle r, \pi \rangle = \text{trace} \rho_\pi(r),$$

then

$$(5) \quad \text{trace}(A(r)I(f)) = \sum_{\pi} \langle r, \pi \rangle \text{trace } \pi(f).$$

3.2. Standard intertwining operators. Let σ be an irreducible admissible unitary representation of M . For $w \in W(\sigma)$, take a representative $\bar{w} \in K \subset G$, a good maximal compact subgroup. Let $\mathbf{A}(\nu, \sigma, \bar{w})$, $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, be the standard intertwining operator defined in Section 1 of [Sha2]. (For detailed description, see [Sha1].) Then $\nu \mapsto \mathbf{A}(\nu, \sigma, \bar{w})$ is a meromorphic function of ν . If $\mathbf{A}(\nu, \sigma, \bar{w})$ is holomorphic at $\nu = 0$, then $\mathbf{A}(0, \sigma, \bar{w})$ is an intertwining operator between $i_{G,M}(\sigma)$ and $i_{G,M}(\bar{w}\sigma)$. We define a normalized intertwining operator

$$A'(\nu, \sigma, \bar{w}) = n(\nu, \sigma, \bar{w})\mathbf{A}(\nu, \sigma, \bar{w}),$$

where $n(\nu, \sigma, \bar{w})$ is a normalizing factor. We do not specify the normalizing factor used here, we just refer to [Art2], Theorem 2.1, for the proof of existence. Set $A'(\sigma, \bar{w}) = A'(0, \sigma, \bar{w})$. One of the basic properties of normalized operators is

$$(6) \quad A'(\sigma, \bar{w}_1\bar{w}_2) = A'(\bar{w}_1\sigma, \bar{w}_2)A'(\sigma, \bar{w}_2),$$

where \bar{w}_1, \bar{w}_2 are representatives of $w_1, w_2 \in W(\sigma)$. Let $\langle \cdot, \cdot \rangle$ be the pairing explained in Section 2 of [Sha1] (also, cf. Section 5.2.1 of [Sil3] and Section 1 of [Art2]). Then the adjoint operator of $A'(\sigma, \bar{w})$ is $A'(\bar{w}\sigma, \bar{w}^{-1})$, i.e., for $f \in i_{G,M}(\sigma)$ and $f' \in i_{G,M}(\bar{w}\sigma)$, we have

$$\langle A'(\sigma, \bar{w})f, f' \rangle = \langle f, A'(\bar{w}\sigma, \bar{w}^{-1})f' \rangle.$$

Then, using (6),

$$\langle A'(\sigma, \bar{w})f, A'(\sigma, \bar{w})f \rangle = \langle f, A'(\bar{w}\sigma, \bar{w}^{-1})A'(\sigma, \bar{w})f \rangle = \langle f, f \rangle.$$

This tells us that $A'(\nu, \sigma, \bar{w})$ is holomorphic at 0 and $A'(\sigma, \bar{w}) \in \text{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\bar{w}\sigma))$.

Next, we can associate to $A'(\sigma, \bar{w})$ an operator in $C(\sigma)$, as follows. Since $w\sigma \cong \sigma$, σ extends to a representation σ_w of the smallest group containing M and \bar{w} . Fix such an extension σ_w and define

$$(7) \quad A(\sigma, w) = \sigma_w(\bar{w})A'(\sigma, \bar{w}).$$

Then $A(\sigma, w) \in C(\sigma)$ and the definition is independent of the representative \bar{w} . Note that $\sigma_w(\bar{w}) \in \text{Hom}_M(\bar{w}\sigma, \sigma)$. Let $w_1, w_2 \in W(\sigma)$, with representatives \bar{w}_1, \bar{w}_2 . Since σ is irreducible, there exists a constant $\eta(w_1, w_2)$ such that

$$\sigma_{w_1 w_2}(\bar{w}_1 \bar{w}_2) = \eta(w_1, w_2) \sigma_{w_1}(\bar{w}_1) \sigma_{w_2}(\bar{w}_2).$$

It follows that

$$(8) \quad A(\sigma, w_1 w_2) = \eta(w_1, w_2) A(\sigma, w_1) A(\sigma, w_2).$$

Now, let R be a subgroup of $W(\sigma)$. Equation (8) implies

$$(9) \quad A(\sigma, r_1 r_2) = \eta(r_1, r_2) A(\sigma, r_1) A(\sigma, r_2), \quad r_1, r_2 \in R.$$

We will show that (3) of Properties 1.1 hold for R .

If $\eta = 1$, then $r \mapsto A(\sigma, r)$ is a representation of R on $C(\sigma)$ and (5) follows directly. Assume η splits, i.e., there exists a function $\xi : R \rightarrow \mathbb{C}^\times$ such that

$$\eta(r_1, r_2) = \xi(r_1 r_2) \xi(r_1)^{-1} \xi(r_2)^{-1}.$$

Then $\xi(r_1 r_2)^{-1} A(\sigma, r_1 r_2) = \xi(r_1)^{-1} A(\sigma, r_1) \xi(r_2)^{-1} A(\sigma, r_2)$ and $r \mapsto \xi(r)^{-1} A(\sigma, r)$ is a representation of R on $C(\sigma)$. Formula (5) holds for $\xi(r)^{-1} A(\sigma, r)$ (noting that in this case, the cocycle is normally absorbed into the normalization so does not appear).

If η does not split, we apply Schur's theory of projective representations ([C-R], §53, [Sch], [Art4]). Theorem 53.3 in [C-R] tells us the class $\{\eta\}$ has finite order n . From the proof of the same theorem, the class $\{\eta\}$ has a representative α whose values $\alpha(r, s)$ are n th roots of 1 and $\alpha(1, 1) = 1$; $\alpha \in \{\eta\}$ means there exists a function $\xi : R \rightarrow \mathbb{C}^\times$ such that

$$\eta(r_1, r_2) = \alpha(r_1, r_2) \xi(r_1 r_2) \xi(r_1)^{-1} \xi(r_2)^{-1}.$$

Fix ζ , a primitive n th root of 1. For each pair $r, s \in R$, define the integer $a_{r,s}$ by

$$\alpha(r, s) = \zeta^{a_{r,s}},$$

$0 \leq a_{r,s} < n$. Let Z_α be the cyclic group generated by α . On the set \tilde{R} of all ordered pairs (r, α^k) , $r \in R$, $\alpha^k \in Z_\alpha$, define multiplication by

$$(r, \alpha^k)(s, \alpha^l) = (rs, \alpha^{a_{r,s}+k+l}).$$

Then \tilde{R} is a group. (To see associativity, observe that (9) for $A'(r) = \xi^{-1}(r)A(\sigma, r)$ gives $A'(r_1 r_2) = \alpha(r_1, r_2)A'(r_1)A'(r_2)$. Associativity follows from $\alpha(r_1 r_2, r_3)\alpha(r_1, r_2) = \alpha(r_1, r_2 r_3)\alpha(r_2, r_3)$, which can be obtained easily from (9) for $A'(r_1 r_2 r_3)$.) The mapping $z \mapsto (1, z)$ is an isomorphism of Z_α into the center of \tilde{R} . We have

$$1 \rightarrow Z_\alpha \rightarrow \tilde{R} \rightarrow R \rightarrow 1.$$

Define a linear character $\chi : Z_\alpha \rightarrow \mathbb{C}^\times$ by $\chi(\alpha^k) = \zeta^k$. Define $\tilde{A} : \tilde{R} \rightarrow C(\sigma)$ by

$$\tilde{A}((r, z)) = \chi^{-1}(z) \xi(r)^{-1} A(\sigma, r),$$

$r \in R$, $z \in Z_\alpha$. Then

$$\begin{aligned} \tilde{A}((r, \alpha^k)(s, \alpha^l)) &= \tilde{A}((rs, \alpha^{a_{r,s}+k+l})) \\ &= \zeta^{-a_{r,s}-k-l} \xi(rs)^{-1} \eta(r, s) A(\sigma, r) A(\sigma, s) \\ &= \zeta^{-a_{r,s}-k-l} \xi(rs)^{-1} \alpha(r, s) \xi(rs) \xi(r)^{-1} \xi(s)^{-1} A(\sigma, r) A(\sigma, s) \\ &= \zeta^{-a_{r,s}-k-l} \zeta^{a_{r,s}} \xi(r)^{-1} \xi(s)^{-1} A(\sigma, r) A(\sigma, s) \\ &= \tilde{A}((r, \alpha^k)) \tilde{A}((s, \alpha^l)). \end{aligned}$$

It follows that $\tilde{r} \mapsto \tilde{A}(\tilde{r})$, $\tilde{r} \in \tilde{R}$, is a homomorphism of \tilde{R} into $C(\sigma)$ and formula (5) holds for \tilde{A} . In particular, attached to each component π is a representation ρ_π of the group \tilde{R} . For $r \in R$, define

$$\langle r, \pi \rangle = \text{trace } \rho_\pi((r, 1)).$$

Lemma 3.1 now implies the following:

THEOREM 3.2. *With notation as above,*

$$\xi^{-1}(r) \text{trace}(A(\sigma, r)I(f)) = \sum_{\pi} \langle r, \pi \rangle \text{trace } \pi(f).$$

3.3. Arthur R -group. We now consider the situation described in Section 1.2. In particular, ψ is an elliptic A -parameter of M and σ belongs to the A -packet $\Pi_\psi(M)$. The group S_ψ (the centralizer in \hat{G} of the image of ψ) is a reductive group. It can be shown that W_ψ/W_ψ^0 is isomorphic to a subgroup of W_ψ ; denote this subgroup by R_ψ . In addition, it can be shown that the following exact sequence splits:

$$1 \longrightarrow W_\psi^0 \longrightarrow W_\psi \longrightarrow R_\psi \longrightarrow 1.$$

Then we have an embedding

$$\begin{array}{ccccccc} 1 & \longrightarrow & W_{\psi,\sigma}^0 & \longrightarrow & W_{\psi,\sigma} & \longrightarrow & R_{\psi,\sigma} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & W_\psi^0 & \longrightarrow & W_\psi & \longrightarrow & R_\psi \longrightarrow 1 \end{array}$$

of short exact sequences. If $R_{\psi,\sigma} \subset W(\sigma)$, we can apply results of sections 3.1 and 3.2 to $R = R_{\psi,\sigma}$.

EXAMPLE 3.3. *We look at what happens in the example from chapter 2. In this case, $\pi = \text{Ind}_P^G(\sigma) = \text{St}_{GL(2)} \times \text{triv}_{GL(2)} \rtimes 1$ and $W(\sigma) = \{1, w_1, w_2, w_3\}$, where the elements of $W(\sigma)$ are most easily described by their actions: if $m = \text{diag}(X_1, X_2, 1, {}^\tau X_2^{-1}, {}^\tau X_1^{-1}) \in M$ (block diagonal matrix in $SO(9, F)$), then $w_1 \cdot m = \text{diag}(X_1, {}^\tau X_2, 1, X_2, {}^\tau X_1)$, $w_2 \cdot m = \text{diag}({}^\tau X_1, X_2, 1, {}^\tau X_2, X_1)$, and $w_3 \cdot m = \text{diag}({}^\tau X_1, {}^\tau X_2, 1, X_2, X_1)$.*

Recall that we have (cf. proof of Theorem 2.5)

$$\begin{aligned} \text{St}_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) &= L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1) \oplus L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2) \\ \text{St}_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; \text{St}_{SO(3)}) &= L(\nu^{-\frac{1}{2}}; \mathcal{S}). \end{aligned}$$

We may also determine that

$$\begin{aligned} \text{triv}_{GL(2)} \rtimes \mathcal{T}_1 &= L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1) \\ \text{triv}_{GL(2)} \rtimes \mathcal{T}_2 &= L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2) \oplus L(\nu^{-\frac{1}{2}}; \mathcal{S}) \end{aligned}$$

by a similar calculation (though made much easier since the Jacquet modules of the various subquotients are already known).

We know $\text{St}_{GL(2)} \rtimes 1 = \mathcal{T}_1 \oplus \mathcal{T}_2$ and $\text{triv}_{GL(2)} \rtimes 1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) \oplus L(\nu^{-\frac{1}{2}}; \text{St}_{SO(3)})$ (cf. Lemma 2.3). We use normalizations so that the nontrivial normalized standard intertwining operator for $\text{St}_{GL(2)} \rtimes 1$ acts trivially on \mathcal{T}_2 (generic component) and nontrivially on \mathcal{T}_1 ; for $\text{triv}_{GL(2)} \rtimes 1$, we normalize so that the nontrivial normalized standard intertwining operator acts trivially on $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ (K -spherical component) and nontrivially on $L(\nu^{-\frac{1}{2}}; \text{St}_{SO(3)})$. We then use corresponding normalizations for $\text{St}_{GL(2)} \times \text{triv}_{GL(2)} \rtimes 1$ (so that $A(\sigma, w_1)$ acts trivially on $\text{St}_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ and nontrivially on $\text{St}_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; \text{St}_{SO(3)})$, etc.). The action of normalized standard intertwining operators is summarized below:

	$A(\sigma, 1)$	$A(\sigma, w_1)$	$A(\sigma, w_2)$	$A(\sigma, w_3)$
$\pi_1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1)$	1	1	1	1
$\pi_2 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2)$	1	1	-1	-1
$\pi_3 = L(\nu^{-\frac{1}{2}}, \mathcal{S})$	1	-1	1	-1

If we let ρ_0 denote the trivial character of $R = W(\sigma)$ and ρ_i for $i = 1, 2, 3$ the nontrivial character which is trivial on w_i , then the above implies $\rho_{\pi_1} = \rho_1$, $\rho_{\pi_2} = \rho_0$ and $\rho_{\pi_3} = \rho_2$ (with $m_{\rho_4 \otimes \pi} = 0$ for all π). If we twist the normalizations by a character ρ of R , there is a corresponding twist in the ρ_{π_i} .

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