ARTHUR R-GROUPS, CLASSICAL R-GROUPS, AND AUBERT INVOLUTIONS FOR $SO(2n+1)$

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ABSTRACT. For the special orthogonal group $G = SO(2n+1)$ over a p-adic field, we consider a discrete series representation of a standard Levi subgroup of G. We prove that the Arthur R-group and the classical R-group of π are isomorphic. If π is generic, we consider the Aubert involution $\hat{\pi}$. Under the assumption that $\hat{\pi}$ is unitary, we prove that the Arthur R-group of $\hat{\pi}$ is isomorphic to the R-group of $\hat{\pi}$ defined in [5, 6]. This is done by establishing the connection between the A-parameters of π and $\hat{\pi}$. We prove that the A-parameter of $\hat{\pi}$ is obtained from the A-parameter of π by interchanging the two $SL(2,\mathbb{C})$ components.

1. Introduction

Let G be a connected reductive quasi-split algebraic group defined over a p-adic field F. Let M be a Levi subgroup of a parabolic subgroup P of G defined over F.

Suppose π is a discrete series representation of $M(F)$, and $I(\pi) = Ind_{P(F)}^{G(F)}\pi$ the representation of $G(F)$ parabolically induced from π. The classical R-group $R(\pi)$ associated to π has been defined for studying the irreducible composition factors of $I(\pi)$. The R-group is a subquotient of the Weyl group and the normalized intertwining operators, corresponding to elements of $R(\pi)$, form a basis of the commuting algebra $Hom_{G(F)}(I(\pi), I(\pi))$ [31]. In essence, $R(\pi)$ is characterized by Plancherel measures of π. The R-group can also be defined in terms of the L-group and Langlands' correspondence. In this context, Arthur proposed a conjectural description of R-groups for some nontempered unitary representations. On the other hand, the

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classical definition of the R-group can also be extended to some nontempered unitary representations. Suppose π is a unitary representation such that the Aubert involution $\hat{\pi}$ is discrete series. According to [5, 6], it is natural to define $R(\pi) = R(\hat{\pi})$, and the R-group $R(\pi)$ has the right basic properties. We call $R(\pi)$ the classical R-group of π.

Let W_F be the Weil group of F, and $W_F' = W_F \times SL_2(\mathbb{C})$ be the Weil-Deligne group [34]. Let ${}^LG = {}^LG^o \times W_F$ be the L-group of G, c.f [8]. Langlands defined certain homomorphisms of W_F' to ${}^LG$, called L-parameters. The local Langlands correspondence predicts that the set of equivalence classes of irreducible admissible representations of $G(F)$ can be partitioned into finite sets, called L-packets. Each L-packet should be parametrized by an L-parameter of W_F', in accordance with the natural functorial principle (see [1]). This is another aspect of the decomposition of a parabolically induced representation and it is natural to ask what is the corresponding aspect of the classical R-group. Arthur outlined the answer in [1]. To deal with nontempered representations, he first extended the definition of L-parameters to Arthur parameters (or briefly A-parameters). Then he defined the R-group associated to an A-parameter and a representation in the Arthur packet of this A-parameter. Arthur expects that his R-groups should be related intimately to the classical R-groups. For example, when the L-packet corresponding to the parameter consists of discrete series representations, then these two R-groups should be the same. This was proved by Shelstad [29], for real groups, and by Keys [18], when M is a maximal torus over F and π is a unitary character of $M(F)$. Actually, as long as the Langlands correspondence is established, the equality of R-groups would follow, as one can see from our proof of Theorem 3.1, in the case when G is split over F. Indeed, the Plancherel measures determine the classical R-group of π, while the Plancherel measures are essentially determined by Langlands-Shahidi L-functions. On the other hand, one can associate certain L-functions, called Artin L-functions, to every L-parameter of G (see [34]). Under the Langlands correspondence, the Langlands-Shahidi L-functions should be equal to the corresponding Artin L-functions. So the Arthur R-group of π is expected to be isomorphic to the classical R-group of π. The Langlands correspondence for GL, due to Harris and Taylor [11] and Henniart [12], Henniart [13] and the result of Jiang and Soudry [17] on $SO(2n+1)$, allow us to prove Arthur's expectation for $SO(2n+1)$. This is our Theorem 3.1.

As Arthur pointed out in [1], things seem to get out of control when we step to the non-tempered situation. A significant discovery made by Zelevinsky [35] relates a discrete series representation of GL to the corresponding Langlands quotient. These two representations are associated to each other by Zelevinsky involution. From our point of view, the link is the following: the L-parameter of the former and the A-parameter of the latter have the same image in the L-group. Consequently, the Arthur R-groups for these two representations are equal. Aubert generalized the idea of Zelevinsky and defined a duality operator [4], for any p-adic group. For GL, Zelevinsky built all irreducible admissible representations and defined the Zelevinsky involution for every irreducible admissible representation, by starting with segments. For other classical groups, segments also have fundamental meaning (see Muić [23]). Though the L-parameter and A-parameter of π and its Aubert involution generally do not relate to each other so obviously, there are still important links inside. Moeglin's work [22] already provided some evidence. As our first understanding of these matters, we look in this paper at the simplest case, i.e., the situation when π is a generic discrete series representation (see Theorem 5.3 and Corollary 5.1). To exploit the links between L-parameters and A-parameters of an irreducible admissible representation and its Aubert involution, more general situations will be dealt with in our future work.

Theorem 3.1 and Theorem 5.3 are our main results, and we will state them here. We work on $SO(2n+1)$.

Theorem 1.1. *Let ϕ be an elliptic tempered L-parameter of M and π be an element of the L-packet of ϕ. Assume that all the members of the packet have same Plancherel* *measures. Then the Arthur R-group and the classical R-group of π are isomorphic and depend only on the L-parameter ϕ.*

We should point out that the assumption in the theorem on Plancherel measures for a discrete packet is a conjecture made by Shahidi in [27].

Theorem 1.2. *Let π be a generic discrete series representation of $M(F)$ and $\hat{\pi}$ the Aubert involution of π. Assume $\hat{\pi}$ is unitary. Then the Arthur R-group and the classical R-group of $\hat{\pi}$ are isomorphic.*

Let us mention that the representation $\hat{\pi}$ in Theorem 1.2 is generally non-tempered (see Corollary 4.1). It is tempered only in the case when π is supercuspidal, that is, $\hat{\pi} = \pi$.

As we pointed out in the above, the results on Langlands correspondence of Harris and Taylor [11] and Henniart [12], Henniart [13] and the result of Jiang and Soudry [17] are the main ingredients for Theorem 3.1. This theorem together with the precise expressions of L-parameters of a generic discrete representation and its Aubert involution in Theorem 5.2, implies Theorem 5.3. The computation in Theorem 5.2 of L-parameters is based on the work of Muić [23] and Jiang and Soudry [17].

The proof of Theorem 5.3 is based on the fact that the A-parameters of π and $\hat{\pi}$ have the same image in ${}^LG$. Actually, Corollary 5.1 proves that the A-parameter of $\hat{\pi}$ is obtained from the A-parameter of π by switching the two $SL(2,\mathbb{C})$ components. We state Corollary 5.1 here.

Corollary 1.1. *Let π be a generic discrete series representation of $G(F) = SO(2n+1, F)$. Let $\psi : W_F \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to {}^LG$ be the A-parameter of π and $\hat{\psi}$ the A-parameter of $\hat{\pi}$. Then,*

$$\hat{\psi}(w, x, y) = \psi(w, y, x).$$

In particular, $\hat{\psi}$ and ψ have the same image in ${}^LG$.

This corollary is related to a general conjecture on the action of the Aubert involution on A-packets. Barbasch conjectured that the Aubert involution sends an A-packet to an A-packet. This raises the question of the action of the involution on A-parameters. It is conjectured that the involution acts on A-parameters of G by interchanging two copies of $SL(2,\mathbb{C})$. Although this conjecture was known previously, the precise statement is due to Hiraga [14].

We now give a short summary of the paper. In Section 2, we recall some basic definitions and properties of L-parameters, A-parameters, and Arthur R-groups. In Section 3, we prove Theorem 3.1, after a lemma on the correspondence of Weyl groups and one on Artin L-functions. Langlands data of the Aubert involution of generic representations are described in Section 4. In Section 5, we first compute the L-parameters of generic discrete representations and their Aubert involutions and then prove Theorem 5.3.

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2. L-PARAMETERS AND A-PARAMETERS

In this section, we recall some basic definitions and properties of L-parameters, A-parameters and Arthur R-groups. Our presentation follows Arthur's paper [1].

Let F be a nonarchimedean local field of characteristic zero, W_F the Weil group of F, $W_F' = W_F \times SL_2(\mathbb{C})$ the Weil-Deligne group of F. Let r be a finite dimensional semisimple continuous complex representation of W_F, and V the representation space of r. One associates to r a complex function $L(s, r)$, called Artin L-function, and ϵ-factor $\epsilon(s, r, \psi_F)$ with ψ_F a fixed additive character of F, called Artin ϵ-factor. The Artin L-function is defined by

$$L(s, r) = det(I - r(\Phi_F)q^{-s}\, V^{I_F})^{-1},$$

where Φ_F is a Frobenius element of W_F, I_F is the inertia group of W_F, V^{I_F} is the subspace of V pointwise fixed by the action of I_F, and q is the cardinality of the residue field of F. Every irreducible finite dimensional continuous complex representation of W'_F is of the form of $r \otimes S_n$ with r an irreducible representation of W_F and S_n the n dimensional irreducible complex representation of $SL_2(\mathbb{C})$. One can define the Artin L-function associated to $r \otimes S_n$ by

$$L(s, r \otimes S_n) = L(s + (n-1)/2, r).$$

Let G be a quasi-split connected reductive algebraic group defined over F. We define the L-group by

$${}^LG = {}^LG^o \times W_F,$$

where ${}^LG^o$ is the connected reductive complex group whose root datum is dual to that of G. The action of W_F on ${}^LG^o$ is induced from the action of the Galois group $Gal(\bar{F}/F)$ on G, where $\bar{F}$ is the algebraic closure of F (see [8]).

We fix a maximal torus T and a Borel subgroup B of G containing T, both defined over F. We also fix a maximal torus ${}^LT^o$ and a Borel subgroup ${}^LB^o$ containing ${}^LT^o$ of ${}^LG^o$, both invariant under the action of W_F corresponding to the duality between the root datum of G and ${}^LG^o$. Let $\Sigma(T, G)$ be the roots of T in G, $\Sigma({}^LT^o, {}^LG^o)$ the roots of ${}^LT^o$ in ${}^LG^o$. It is well known that the two Weyl groups are isomorphic, i.e. $W(T, G) \simeq W({}^LT^o, {}^LG^o)$, under the map sending s_α to $s_{\alpha^\vee}$. Here $\alpha^\vee \in \Sigma({}^LT^o, {}^LG^o)$ is the coroot of $\alpha \in \Sigma(T, G)$ and s_α, $s_{\alpha^\vee}$ are the corresponding reflections on the spaces generated by $\Sigma(T, G)$ and $\Sigma({}^LT^o, {}^LG^o)$ respectively. Let T_d be the maximal split torus of G contained in T, and ${}^LT^o_d$ the maximal subtorus pointwise fixed by W_F contained in ${}^LT^o$. Then $W({}^LT^o_d, {}^LG^o)$ is the subgroup of $W({}^LT^o, {}^LG^o)$ consisting of the elements fixed by W_F. So, we have an isomorphism between $W(T_d, G)$ and $W({}^LT^o_d, {}^LG^o)$.

Let M be a Levi subgroup of a standard parabolic subgroup of G defined over F with respect to B, and A the split component of $Z(M)$, the center of M. Let ${}^LM$ denote the L-group of M. Then ${}^LM$ is a Levi subgroup of ${}^LG$, by the bijection [8] between the set of conjugacy classes of the parabolic subgroups over F of G and the set of conjugacy (in ${}^LG^o$) classes of parabolic subgroups of ${}^LG$. Let ${}^LA^o$ be the maximal torus of the center of ${}^LM^o$, and U be the maximal subtorus of ${}^LA^o$ pointwise fixed by W_F. It is well known that $W(A, G)$ can be identified with the subgroup of $W(T_d, G)$ whose elements stabilize the set of positive roots of T_d in M with respect to the Borel subgroup B. Indeed, let $W'(T_d, G)$ be the subgroup of $W(T_d, G)$ consisting of the elements which stabilize $\Sigma(T_d, M)$, the roots of T_d in M. Then $W(T_d, M)$ is a normal subgroup of $W'(T_d, G)$. The group $W(A, G)$ is isomorphic to $W'(T_d, G)/W(T_d, M)$. By Lemma 1.1.2 in [9], every coset of $W'(T_d, G)/W(T_d, M)$ has a unique element which stabilizes the set of positive roots of T_d in M. Let $W^+(T_d, G)$ be the set of all such elements. Then $W^+(T_d, G)$ is the subgroup of $W(T_d, G)$ whose elements stabilize the set of the positive roots of T_d in M. So, $W'(T_d, G) = W(T_d, M) \times W^+(T_d, G)$. Therefore, $W(A, G)$ is identified with $W^+(T_d, G)$. The same is true for $W(U, {}^LG^o)$ in $W({}^LT^o_d, {}^LG^o)$. So the isomorphism between $W(T_d, G)$ and $W({}^LT^o_d, {}^LG^o)$ induces an isomorphism between $W(A, G)$ and $W(U, {}^LG^o)$.

A homomorphism

$$\phi : W_F \times SL_2(\mathbb{C}) \to {}^LG = {}^LG^o \times W_F$$

which commutes with the projections to W_F is called a Langlands parameter, or L-parameter of G, if the conditions 1)-5) on pages 41-42 of [20] are satisfied by ϕ.

The parameter ϕ is elliptic, if its image is not contained in any proper Levi subgroup of ${}^LG$. It is tempered, if the image of the projection of $\phi(W_F)$ to ${}^LG^o$ is bounded. We say that two L-parameters ϕ and ϕ' are equivalent, and write $\phi \sim \phi'$, if they are conjugate in ${}^LG^o$. We denote by $\Phi(G)$ the set of equivalence classes of L-parameters of G.

Let π be an irreducible admissible representation of $G(F)$ and r be a continuous finite dimensional semisimple complex representation of ${}^LG$. There are a conjectural L-function $L(s, \pi, r)$ and ϵ-factor $\epsilon(s, \pi, r, \psi_F)$ attached to π and r. Here ψ_F is a fixed additive character of F. Denote by $\Pi(G)$ the set of equivalence classes of irreducible admissible representations of $G(F)$. Langlands correspondence predicts that $\Pi(G)$ can be partitioned into finite sets, called L-packets, such that there is a bijection between the set of L-packets of G and $\Phi(G)$. Let $\Pi_\phi(G)$ be the L-packet of G corresponding to an L-parameter ϕ. Then, for any r as above and any $\pi \in \Pi_\phi(G)$, one should have the equalities:

$$L(s, r \circ \phi) = L(s, \pi, r),$$

$$\epsilon(s, r, \psi_F) = \epsilon(s, \pi, r, \psi_F).$$

It is expected that the L-packet of a tempered L-parameter consists of tempered representations, that of an elliptic L-parameter consists of discrete series representations.

To deal with Langlands quotients, Arthur extended the definition of L-parameters to a kind of more general parameters called Arthur parameters, or A-parameters. A homomorphism

$$\psi : W'_F \times SL_2(\mathbb{C}) \to {}^LG$$

commuting with the projections on W_F is called an A-parameter, if

(1) the restriction of ψ to W'_F is tempered,

(2) ψ is also analytic on the second copy of $SL_2(\mathbb{C})$.

The definitions of elliptic A-parameters and equivalent A-parameters are similar to those of L-parameters. We shall write $\Psi(G)$ for the set of equivalence classes of A-parameters of G. It is expected that, like L-parameters, every A-parameter ψ

determines a finite set $\Pi_\psi(G)$ in $\Pi(G)$, and the map

$$\psi \mapsto \Pi_\psi(G)$$

satisfies some nice properties (see [1]). $\Pi_\psi(G)$ is called the Arthur packet (or A-packet) associated to ψ. We should remind that, unlike L-packets, two A-packets of two distinct A-parameters may have a non-empty intersection. Tempered L-parameters are naturally embedded in $\mathbf{\Psi}(G)$: a tempered L-parameter ϕ becomes an A-parameter if we let $\phi(w,x,y) = \phi(w,x)$. And for $\psi \in \mathbf{\Psi}(G)$, Arthur defined an L-parameter ϕ_ψ by letting

$$\phi_\psi(w,x) = \psi(w,x,\begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}) \tag{1}$$

for every $(w,x) \in W_F'$. The map $\psi \mapsto \phi_\psi$ is injective, see Proposition 1.3.1 in [3]. Suppose ϕ is the L-parameter of the L-packet containing π. Then ψ is called the A-parameter of π if ϕ_ψ is the L-parameter of π, i.e., $\phi_\psi = \phi$. It is expected that $\Pi_\phi(G) \subset \Pi_\psi(G)$ [1].

Suppose that ψ is an A-parameter which factors through a Levi subgroup ${}^LM = {}^LM^o \times W_F$ of ${}^LG$, but not any proper Levi subgroup of ${}^LM$. Since G is quasi-split over F, there is a Levi subgroup M of G defined over F such that ${}^LM$ is the L-group of M. Therefore, ψ is an elliptic A-parameter of M. Arthur associated to ψ several related groups. Let S_ψ be the centralizer in ${}^LG^o$ of the image of ψ, and S_ψ^o the identity component of S_ψ, T_ψ a maximal torus of S_ψ^o. Define

$$W_\psi = N_{S_\psi}(T_\psi)/Z_{S_\psi}(T_\psi),$$

$$W_\psi^o = N_{S_\psi^o}(T_\psi)/Z_{S_\psi^o}(T_\psi),$$

$$R_\psi = W_\psi/W_\psi^o.$$

Here we write $N_G(H)$ for the normalizer of H in G, and $Z_G(H)$ for the centralizer of H in G. W^o_ψ is normal in W_ψ, since S^o_ψ is normal in S_ψ. The following Lemma 2.2 tells us that W_ψ is a subgroup of $W(U, {}^LG^o)$.

Lemma 2.1. *Suppose that*

$$\psi : W'_F \times SL_2(\mathbb{C}) \to {}^LG$$

is an A-parameter. Then

1. *$Z_{^LG}(T_\psi)$ is a Levi subgroup of ${}^LG$;*
2. *Let ${}^LM = Z_{^LG}(T_\psi)$. Then ψ is an elliptic A-parameter of M.*

Proof. Since the projection of $Z_{^LG}(T_\psi)$ to W_F is onto, $Z_{^LG}(T_\psi)$ is a Levi subgroup of ${}^LG$, by Borel's Lemma 3.5 in [8].

Let ${}^LM = Z_{^LG}(T_\psi)$, where M is a Levi subgroup of G over F. Then the image of ψ is contained in ${}^LM$. Suppose that ${}^LM_o$ is a Levi subgroup of ${}^LM$ also containing the image of ψ. By the same lemma of Borel, ${}^LM_o = Z_{^LG}(S)$, where S is a torus of ${}^LG^o$. Since the image of ψ is contained in ${}^LM_o$, it follows that S is contained in S^o_ψ. So, S is contained in a conjugate of T_ψ in S^o_ψ. We have ${}^LM \subset s_o^{-1}{}^LM_o s_o$, for an $s_o \in S^o_\psi$. ${}^LM_o \subset {}^LM$ implies ${}^LM_o = {}^LM$. Therefore, ψ is elliptic for M. $\square$

We denote by $W_\psi({}^LA^o, {}^LG^o)$ the subgroup of $W({}^LA^o, {}^LG^o)$ consisting of the elements which can be represented by elements of S_ψ.

Lemma 2.2. *Let M be a Levi subgroup of G defined over F, ψ be an elliptic A-parameter of M. Suppose that ${}^LA^o$ is the maximal torus of the center of ${}^LM^o$ and U is the maximal subtorus of ${}^LA^o$ pointwise fixed by W_F. Then*

1. *$({}^LA^o \cap S_\psi)^o$ is a maximal torus of S^o_ψ;*
2. *${}^LA^o \cap S_\psi = ({}^LA^o)^{W_F}$, (the W_F fixed points of ${}^LA^o$), and $U = ({}^LA^o \cap S_\psi)^o$;*
3. *W_ψ can be identified with $W_\psi({}^LA^o, {}^LG^o)$.*

Proof. (1) Let T_ψ be a maximal torus of S_ψ^o, and ${}^LM_o = Z_{{}^LG}(T_\psi)$. Then Lemma 2.1 says that ψ is elliptic for ${}^LM_o$. From Proposition 3.6 of [8], there is $s_o \in S_\psi^o$ such that $s_o^{-1}\,{}^LM_o s_o = {}^LM$. So, ${}^LM = Z_{{}^LG}(s_o^{-1}T_\psi s_o)$. Hence, ${}^LM^o = Z_{{}^LG^o}(s_o^{-1}T_\psi s_o)$. Since $s_o^{-1}T_\psi s_o$ is contained in the center of ${}^LM^o$, $s_o^{-1}T_\psi s_o$ is contained in ${}^LA^o$. We claim $s_o^{-1}T_\psi s_o = ({}^LA^o \cap S_\psi)^o$. In fact, $s_o^{-1}T_\psi s_o \subset {}^LA^o \cap S_\psi^o$ implies $s_o^{-1}T_\psi s_o \subset ({}^LA^o \cap S_\psi^o)^o = ({}^LA^o \cap S_\psi)^o$. Since $s_o^{-1}T_\psi s_o$ is also a maximal torus of S_ψ^o, we have $s_o^{-1}T_\psi s_o = ({}^LA^o \cap S_\psi)^o$. Therefore, $({}^LA^o \cap S_\psi)^o$ is a maximal torus of S_ψ^o.

(2) We write $\psi(w,x,y) = g(w,x,y)w$ with $g(w,x,y) \in {}^LG^o$ and $w \in W_F$, for $(w,x,y) \in W_F' \times SL_2(\mathbb{C})$. For any $a \in {}^LA^o \cap S_\psi$, we have

$$ag(w,x,y)w = g(w,x,y)wa = g(w,x,y)a^w w.$$

Therefore, $a \in {}^LA^o$ implies

$$g(w,x,y)aw = g(w,x,y)a^w w, \text{ for } (w,x,y) \in W_F' \times SL_2(\mathbb{C}) .$$

It follows $a \in ({}^LA^o)^{W_F}$, thus ${}^LA^o \cap S_\psi \subset ({}^LA^o)^{W_F}$. On the other hand, $({}^LA^o)^{W_F} \subset {}^LA^o \cap S_\psi$. Therefore, ${}^LA^o \cap S_\psi = ({}^LA^o)^{W_F}$. This implies $U = ({}^LA^o \cap S_\psi)^o$.

(3) Now, we let $T_\psi = ({}^LA^o \cap S_\psi)^o$, a maximal torus of S_ψ^o. We set the map

$$f : W_\psi({}^LA^o, {}^LG^o) \to W(T_\psi, S_\psi)$$

as follows: if ω is represented by $s \in S_\psi$, we define $f(\omega)$ to be the coset of s in $W(T_\psi, S_\psi)$. First, we have to show f is well defined. Note that s normalizes ${}^LA^o \cap S_\psi$. It follows that s normalizes $({}^LA^o \cap S_\psi)^o$ and $s \in N_{S_\psi}(T_\psi)$. Therefore, $f(\omega) \in W(T_\psi, S_\psi)$. Suppose that $\omega = 1$ in $W_\psi({}^LA^o, {}^LG^o)$. Then $s \in {}^LM^o$, hence $s \in Z_{S_\psi}(T_\psi)$. Therefore $f(\omega) = 1$. So far, we have proved f is well defined. Observe that f is a homomorphism of groups. For isomorphism, suppose that $f(\omega) = 1$. Then $s \in Z_{{}^LG^o}(T_\psi) = {}^LM^o$, so $s \in Z_{{}^LG^o}({}^LA^o)$ and $\omega = 1$. This proves that f is injective. Let $s \in N_{S_\psi}(T_\psi)$. We have

$$N_{{}^LG^o}(T_\psi) \subset N_{{}^LG^o}({}^LM^o) = N_{{}^LG^o}({}^LA^o).$$

This implies that $s \in N_{^LG^o}(^LA^o) \cap S_\psi$, so f is surjective. Therefore, f is an isomorphism of the groups. $\square$

Remarks. Let ψ be an elliptic A-parameter of M.

(1) When G is split over F, $^LA^o$ is a maximal torus of S_ψ^o.
(2) W_ψ is a subgroup of $W(U, {}^LG^o)$, from (1) and (2) of Lemma 2.2. And $W_\psi({}^LA^o, {}^LG^o)$ can be identified with a subgroup of $W(U, {}^LG^o)$, from (3) of Lemma 2.2.

$W(A, G)$ acts on the set of isomorphic classes of irreducible admissible representations of $M(F)$. From Lemma 6.2 of [8], every element of $W(U, {}^LG^o)$ can be represented by an element of $N_{^LG^o}(U)$ fixed by W_F. Therefore, $W(U, {}^LG^o)$ acts on $\Psi(M)$. Suppose that ψ is an elliptic A-parameter of M, and $\pi \in \Pi_\psi(M)$. We define

$$W_{\psi,\pi} = \{\omega \in W_\psi \,;\, \omega\pi \simeq \pi\}$$

$$W_{\psi,\pi}^o = \{\omega \in W_\psi^o \,;\, \omega\pi \simeq \pi\}$$

$$R_{\psi,\pi} = W_{\psi,\pi}/W_{\psi,\pi}^o.$$

Lemma 2.3. *Let ψ be an elliptic A-parameter of M. Then*

$$W_\psi = \{\omega \in W(U, {}^LG^o) \,;\, \omega\psi \sim \psi \;\; in \;\; {}^LM^o\}.$$

Proof. Note that

$$W_\psi \subset \{\omega \in W(U, {}^LG^o) \,;\, \omega\psi \sim \psi \;\; in \;\; {}^LM^o\}.$$

Let $\omega \in W(U, {}^LG^o)$ such that $\omega\psi \sim \psi$ in ${}^LM^o$. Suppose that ω is represented by an element n in $N_{^LG^o}(U)$ and n is fixed by W_F. Then there is $m \in {}^LM^o$ depending on n such that $n^{-1}\psi(w, x, y)n = m^{-1}\psi(w, x, y)m$, for every $(w, x, y) \in W_F' \times SL_2(\mathbb{C})$. Therefore $mn^{-1} \in S_\psi$. So $\omega \in W_\psi$. $\square$

Now suppose that G is split over F. Let $\beta \in \Sigma_r(A, P)$. We denote by A_β the maximal subtorus of A contained in the kernel of β, and $M_\beta = Z_G(A_\beta)$. Set $P_\beta = MN_\beta$, where $N_\beta = M_\beta \cap N$. Then M_β is a Levi subgroup of G over F and P_β is a maximal parabolic subgroup of M_β over F with a Levi subgroup M. Let ${}^LM_\beta$ and ${}^LP_\beta$

be the L-groups of M_β and P_β, respectively. Then ${}^LM_\beta$ is a Levi subgroup of ${}^LG$, ${}^LP_\beta$ is a maximal parabolic subgroup of ${}^LM_\beta$ and ${}^LM$ is a Levi subgroup of ${}^LP_\beta$. The set $\Sigma({}^LA^o, {}^LP^o_\beta)$ has a unique reduced root, and we denote this unique reduced root by $\beta^\vee$. We have ${}^LM^o_\beta = ({}^LM^o)_{\beta^\vee}$, where $({}^LM^o)_{\beta^\vee} = Z_{{}^LG^o}(({}^LA^o)_{\beta^\vee})$ and $({}^LA^o)_{\beta^\vee} = (ker(\beta^\vee))^o$.

Lemma 2.4. *Suppose that G is split over F. Let $\beta \in \Sigma_r(A, G)$. Then,*

$$\eta(W(A, M_\beta)) = W({}^LA^o, {}^LM^o_\beta).$$

Proof. We first note the fact that for $\omega \in W(T, G)$, $\alpha \in \Sigma(T, G)$, we have $\eta(\omega)(\alpha^\vee) = (\omega(\alpha))^\vee$. Indeed, when $\omega = s_\gamma$, $\gamma \in \Sigma(T, G)$, this can be verified by computing $< \chi, (s_\gamma(\alpha))^\vee > = < \chi, s_{\gamma^\vee}(\alpha^\vee) >$, for any $\chi \in X(T)$. Here $< \cdot, \cdot >$ is the duality between $X(T)$ and $X_*(T)$, which are characters and co-characters of T, and we identify $X(T)$ and $X_*(T)$ with $X_*({}^LT^o)$ and $X({}^LT^o)$, respectively. For general $\omega \in W(T, G)$, the fact can be proved by induction on the length of ω.

Observe that $\eta(W(T, M)) = W({}^LT^o, {}^LM^o)$, and $\eta(W(T, M_\beta)) = W({}^LT^o, {}^LM^o_\beta)$. $W(A, M_\beta)$ (respectively, $W({}^LA^o, {}^LM^o_\beta)$) is the subgroup of $W(T, M_\beta)$ (respectively, $W({}^LT^o, {}^LM_\beta)$) whose elements stabilize the set of positive roots of $\Sigma(T, M)$ (respectively, $\Sigma({}^LT^o, {}^LM^o)$). So, the fact above implies $\eta(W(A, M_\beta)) = W({}^LA^o, {}^LM^o_\beta)$. $\square$

It is known that $W(A, M_\beta)$ has order one or two. This lemma says that $W(A, M_\beta)$ has order two if and only if $W({}^LA^o, {}^LM^o_\beta)$ has order two. When $W(A, M_\beta)$ has order two, we denote the non-trivial elements of $W(A, M_\beta)$ and $W({}^LA^o, {}^LM^o_\beta)$ by s_β and $s_{\beta^\vee}$, respectively.

3. R-groups for discrete series representations of $SO(2n+1)$

From this section, $G = G_n = SO(2n+1)$ over F, the odd special orthogonal group. Then ${}^LG^o = Sp_{2n}(\mathbb{C})$. Since G is split over F, W_F acts on ${}^LG^o$ trivially.

We realize G as a closed subgroup of $GL(2n+1)$ as the following. Let J_n be the n by n matrix whose entries of the second diagonal are 1 and the other entries are zero. Then

$$G = \{g \in GL(2n+1);\ {}^t g J_{2n+1} g = J_{2n+1}, det(g) = 1\}.$$

Let T be the maximal torus of G consisting of elements

$$x = diag(x_1, \cdots, x_n, 1, x_n^{-1}, \cdots, x_1^{-1});\ x_i \in GL(1).$$

T is defined over F. Let B be the set of upper-triangular matrices in G. Then B is a Borel subgroup of G over F, and $B = TU$, where U is the unipotent radical of B. Let e_i be the character of T sending x to x_i. Then the root basis with respect to T and B is $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, with $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = e_n$. Let P be a standard parabolic subgroup of G over F with respect to B, and $P = MN$ the Levi decomposition of P, where

$$\begin{aligned} M &= \{diag(x_1, \cdots, x_r, x_0, {}^\tau x_r^{-1}, \cdots, {}^\tau x_1^{-1});\ x_i \in GL_{m_i}, x_0 \in G_{m_0}\} \\ &\simeq GL_{m_1} \times \cdots \times GL_{m_r} \times G_{m_0}, \quad \sum_{0 \leq i \leq r} m_i = n. \end{aligned}$$

Here ${}^\tau x_i$ is the transpose of x_i with respect to the second diagonal, and we allow the situation of $m_0 = 0$. The maximal subtorus of $Z(M)$ is

$$A = \{x = diag(x_1 I_{m_1}, \cdots, x_r I_{m_r}, I_{2m_0+1}, x_r^{-1} I_{m_r} \cdots, x_1^{-1} I_{m_1});\ x_i \in GL(1)\}.$$

Let E_i be the character of A sending x to x_i for $1 \leq i \leq r$. Then

$$\Sigma_r(A, P) = \{E_i \pm E_j;\ \ 1 \leq i < j \leq r\} \cup \{E_i; 1 \leq i \leq r\}.$$

Now we consider the L-group of G. We realize ${}^L G^o$ as a closed subgroup of $GL_{2n}(\mathbb{C})$ by letting

$${}^L G^o = \{g \in GL_{2n}(\mathbb{C});\ {}^t g J'_{2n} g = J'_{2n}, \},$$

where $J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$. Fix a maximal torus ${}^L T^o$ of ${}^L G^o$,

$${}^L T^o = \{diag(x_1, \cdots, x_n, x_n^{-1}, \cdots, x_1^{-1});\ x_i \in GL_1(\mathbb{C})\},$$

and the Borel subgroup ${}^LB^o$ of ${}^LG^o$ containing ${}^LT^o$, which consists of all upper-triangular matrices in ${}^LG^o$. We still use e_i for the character of ${}^LT^o$ sending x to x_i. Then the root base with respect to ${}^LT^o$ and ${}^LB^o$ is $\Delta^\vee = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$, where $\alpha_i^\vee = e_i - e_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n^\vee = 2e_n$. The L-group ${}^LM$ of M is a Levi subgroup of ${}^LG$ and

$$\begin{aligned} {}^LM^o &= \{diag(x_1, \cdots, x_r, x_0, {}^\tau x_r^{-1}, \cdots, {}^\tau x_1^{-1}) ;\ x_i \in GL_{m_i}(\mathbb{C}), 1 \le i \le r, x_0 \in {}^LG^o_{m_0}\} \\ &\simeq GL_{m_1}(\mathbb{C}) \times \cdots \times GL_{m_r}(\mathbb{C}) \times {}^LG^o_{m_0}, \quad \sum_{0 \le i \le r} m_i = n. \end{aligned}$$

Let ${}^LA^o$ be the maximal torus of $Z({}^LM^o)$. We have

$${}^LA^o = \{diag(x_1 I_{m_1}, \cdots, x_r I_{m_r}, I_{2m_0}, x_r^{-1} I_{m_r}, \cdots, x_1^{-1} I_{m_1}) ;\ x_i \in \mathbb{C}^\times\}.$$

We also use E_i for the characters of ${}^LA^o$ sending x to x_i. Then

$$\Sigma_r({}^LA^o, {}^LP^o) = \begin{cases} \{E_i \pm E_j ;\ 1 \le i < j \le r\} \cup \{E_i ;\ 1 \le i \le r\} & \text{if } m_0 \neq 0 \\ \{E_i \pm E_j ;\ 1 \le i < j \le r\} \cup \{2E_i ;\ 1 \le i \le r\} & \text{if } m_0 = 0. \end{cases}$$

For $\beta \in \Sigma_r(A, P)$, it can be verified that

$$\beta^\vee = \begin{cases} E_i \pm E_j, & \text{if } \beta = E_i \pm E_j \text{ for } 1 \le i < j \le r \\ E_i, & \text{if } \beta = E_i, 1 \le i \le r \text{ and } m_0 > 0 \\ 2E_i, & \text{if } \beta = E_i, 1 \le i \le r \text{ and } m_0 = 0. \end{cases}$$

For $\beta \in \Sigma_r(A, P)$, $W(A, M_\beta)$ has order two if and only if $\beta = E_i \pm E_j$ with $m_i = m_j$ for $1 \le i < j \le r$ or $\beta = E_i$ for $1 \le i \le r$.

We now describe the actions of $W(A, G)$ and $W({}^LA^o, {}^LA^o)$ on M and ${}^LM^o$, respectively. Let $\beta \in \Sigma_r(A, P)$ such that $W(A, M_\beta)$ has order two, and let $x \in M$ and $y \in {}^LM^o$. s_β (respectively, $s_{\beta^\vee}$) acts on M (respectively, ${}^LM^o$) by exchanging x_i and x_j (respectively, y_i and y_j) if $\beta = E_i - E_j$, by changing x_i to ${}^\tau x_j^{-1}$ and x_j to ${}^\tau x_i^{-1}$ (respectively, y_i to ${}^\tau y_j^{-1}$ and y_j to ${}^\tau y_i^{-1}$) if $\beta = E_i + E_j$, by changing x_i to ${}^\tau x_i^{-1}$ (respectively, y_i to ${}^\tau y_i^{-1}$) if $\beta = E_i$. $W(A, G)$ is generated by s_β, $\beta \in \Sigma_r(A, G)$ such that $W(A, M_\beta)$ has order two. So, for $\omega \in W(A, G)$, ω acts on M by a permutation p on $x_1, \cdots, x_r$ and changing $x_{i_1}, \cdots, x_{i_k}$ to ${}^\tau x_{i_1}^{-1}, \cdots, {}^\tau x_{i_k}^{-1}$, if and only if $\eta(\omega)$ acts on ${}^LM^o$ by the same permutation p on $y_1, \cdots, y_r$ and changing $y_{i_1}, \cdots, y_{i_k}$ to ${}^\tau y_{i_1}^{-1}, \cdots, {}^\tau y_{i_k}^{-1}$, for $x \in M$, $y \in {}^LM^o$. We record this fact in the following lemma.

Lemma 3.1. *$W({}^LA^o, {}^LG^o)$ acts on ${}^LM^o$ in the same way (in the sense above) as $W(A, G)$ does on M.*

Let π be a discrete series representation of $M(F)$. Set

$$W(\pi) = \{\omega \in W(A, G) \,;\, \omega\pi \simeq \pi\}.$$

For $\beta \in \Sigma_r(A, G)$, we denote by $\mu_\beta(\cdot, \pi)$ the Plancherel measure for β and π, see [30]. β is called a special root of π if $\mu_\beta(0, \pi) = 0$. It is known that the set of special roots of π forms a root system. Suppose that β is a special root for π. Then $W(A, M_\beta)$ has order two and s_β is in $W(\pi)$ [30]. We define $W^o(\pi)$ as the subgroup of $W(\pi)$ generated by s_β, where $\beta \in \Sigma_r(A, P)$ and is special for π. $W^o(\pi)$ is a normal subgroup of $W(\pi)$, since $\mu_{\omega\beta}(0, \pi) = \mu_\beta(0, \pi)$ for $\omega \in W(\pi)$. The classical R-group $R(\pi)$ of π satisfies $R(\pi) \simeq W(\pi)/W^o(\pi)$, [31].

Suppose that $\beta \in \Sigma_r(A, P)$. Let ${}^L\mathfrak{g}$ and ${}^L\mathfrak{n}_\beta$ be the Lie algebras of ${}^LG^o$ and ${}^LN^o_\beta$, respectively. Denote by r_β the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}_\beta$. Since G is split over F, W_F acts on ${}^L\mathfrak{g}$ trivially. r_β has a one form of the following forms:

$$r_\beta = \begin{cases} \rho_{m_i} \otimes \tilde{\rho}_{m_j}, \text{ or } \rho_{m_i} \otimes \rho_{m_j}, & 1 \le i < j \le r, \quad \text{or} \\ \rho_{m_i} \otimes \tilde{\rho}'_{m_0} \oplus Sym^2\rho_{m_i}, & 1 \le i \le r, \qquad \text{or} \\ Sym^2\rho_{m_i}, & 1 \le i \le r. \end{cases}$$

The third situation happens only when $m_0 = 0$. Here ρ_m and ρ'_m are the standard representations of $GL_m(\mathbb{C})$ and $Sp_{2m}(\mathbb{C})$ respectively, $Sym^2\rho_m$ is the symmetric square of ρ_m. Shahidi (in [26] and [27]) defined the L-function $L(s, \pi, r_\beta)$ and ϵ-factor for each irreducible generic admissible representation π of $M(F)$ and each r_β for

$\beta \in \Sigma_r(A, P)$. The Plancherel measures are closely related to the L-functions and ϵ-factors, see 3.1 of [26].

For each $\beta^\vee \in \Sigma({}^LA^o, {}^LG^o)$, we define

$$ {}^L\mathfrak{g}_{\beta^\vee} = \oplus\, {}^L\mathfrak{g}_{\alpha^\vee} $$

where $\alpha^\vee$ runs over the roots of ${}^LT^o$ in ${}^LG^o$ restricting to $\beta^\vee$, and ${}^L\mathfrak{g}_{\alpha^\vee}$ is the root space of $\alpha^\vee$ in ${}^L\mathfrak{g}$. Then

$$ {}^L\mathfrak{n}_\beta = \begin{cases} {}^L\mathfrak{g}_{\beta^\vee} \oplus {}^L\mathfrak{g}_{2\beta^\vee} & \text{if } r_\beta = \rho_{m_i} \otimes \tilde{\rho}'_{m_0} \oplus Sym^2\rho_{m_i} \ ; \\ {}^L\mathfrak{g}_{\beta^\vee}, & \text{otherwise.} \end{cases} $$

When $r_\beta = \rho_{m_i} \otimes \tilde{\rho}'_{m_0} \oplus Sym^2\rho_{m_i}$, ${}^L\mathfrak{g}_{\beta^\vee}$ is the space that $\rho_{m_i} \otimes \tilde{\rho}'_{m_0}$ acts on, and ${}^L\mathfrak{g}_{2\beta^\vee}$ is the space that $Sym^2\rho_{m_i}$ acts on. One can see this from [28], or by direct computation.

To continue, we need the following lemma on Artin L-functions. Here we adopt the proof due to Alan Roche, rather than our's which invokes the result of local Langlands correspondence for GL.

Lemma 3.2. *Let $r = r' \otimes S_n$ be an irreducible finite dimensional representation of W'_F with r' bounded. Then the Artin L-function $L(s, r)$ has a pole at $s = 0$ if and only if r is the trivial representation of W'_F, i.e., $n = 1$ and r' is the one dimensional trivial representation of W_F.*

Proof. (Roche) Recall

$$ L(s, r) = L(s + (n-1)/2, r'). $$

Suppose that $L(s, r)$ has a pole at $s = 0$. Let V be the representation space of r'. V^{I_F} is a W_F invariant subspace of V, since I_F is normal in W_F. Since (r', V) is irreducible, $V^{I_F} = 0$ or $V^{I_F} = V$. In the first case, we get $L(s, r') = 1$, hence $L(s, r) = 1$ has no poles. In the second case, (r', V) is an irreducible representation of $W_F/I_F \simeq \mathbb{Z}$. So, r' must be one dimensional and unramified. Since r' is bounded and $L(s + (n-1)/2, r')$ has a pole at $s = 0$, we see that $n = 1$ and r' is the one dimensional trivial representation of W_F.

Conversely, if $n = 1$ and r' is the one dimensional trivial representation of W_F, then $L(s, r) = (1 - q^{-s})^{-1}$ has a pole at $s = 0$. □

At this point, we make a comment on Theorem 2.2 of Jiang and Soudry in [17]. In Theorem 4.2 of [17], they proved that there is a bijection between the set of tempered L-parameters of $SO(2n+1)$ and the set of equivalence classes of generic irreducible tempered representations such that the Rankin-Selberg L-functions and ϵ-factors for $SO(2n+1)$ and GL are preserved. In Theorem 2.2 of [17], they proved that generic discrete series representations correspond to a subset of elliptic tempered L-parameters. We comment that this subset is actually the whole set of elliptic tempered L-parameters. In fact, if ϕ is a tempered L-parameter corresponding to an irreducible generic tempered representation which is not a discrete series representation, then ϕ must factor through a proper Levi subgroup of $Sp_{2n}(\mathbb{C})$, by the construction of such a tempered L-parameter in the proof of Theorem 4.2 of [17].

Theorem 3.1. *Let ϕ be an elliptic tempered L-parameter of M. Assume that all members of $\Pi_\phi(M)$ have same Plancherel measures. Then for any $\pi \in \Pi_\phi(M)$*

(1) $W_\phi = W_{\phi,\pi} \simeq W(\pi)$;
(2) $W_\phi^o = W_{\phi,\pi}^o \simeq W^o(\pi)$;
(3) $R_\phi = R_{\phi,\pi} \simeq R(\pi)$.

Proof. The (3) in the theorem comes from (1) and (2).

Let

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus \phi_0$$

be an elliptic tempered L-parameter of M, and

$$\pi = \pi_1 \otimes \cdots \otimes \pi_r \otimes \pi_0$$

be a discrete series representation of $M(F)$ in the L-packet $\Pi_\phi(M)$.

$W(A,G)$ acts on π by taking a permutation on $\pi_1,\cdots,\pi_r$ and taking some π_i's to their contragredients. So does $W({}^LA^o,{}^LG^o)$ on ϕ. Let $\omega\in W(A,G)$. Then by Lemma 3.1, the permutation of $\eta(\omega)$ on $\phi_1,\cdots,\phi_r$ is the same permutation of ω on $\pi_1,\cdots,\pi_r$, and ω takes π_i to its contragredient if and only if $\eta(\omega)$ takes ϕ_i to its contragredient.

The action of $W(A,G)$ has nothing to do with the L-packet of G_{m_0} containing π_0. Since every L-packet of GL consists of one element, we have that $W(\pi)=W(\pi')$ for π and π' in a same L-packet of M. Therefore, $\eta(\omega)(\phi)\sim\phi$ if and only if $\omega\in W(\pi)$. Lemma 2.3 now implies the (1) of the theorem.

Since all the members of $\Pi_\phi(M)$ have the same Plancherel measures, $W^o(\pi)=W^o(\pi')$ for $\pi,\pi'\in\Pi_\phi(M)$.

Jiang-Soudry's Theorem 2.2 of [17] implies that there is a unique generic member in $\Pi_{\phi_0}(SO(2m_0+1))$. We assume that π_0 is generic. So π is generic, since every discrete series representation of GL is generic. For a root β in $\Sigma_r(A,P)$, $\mu_\beta(0,\pi)=0$ if and only if $L(s,\pi,r_\beta)$ has a pole at $s=0$, [36].

By the results on Langlands correspondence proved by Harris and Taylor [11], Henniart [12] for GL,

$$L(s,\pi,\rho_{m_i}\otimes\tilde\rho_{m_j})=L(s,\rho_{m_i}\otimes\tilde\rho_{m_j}\circ\phi),$$

$$L(s,\pi,\rho_{m_i}\otimes\rho_{m_j})=L(s,\rho_{m_i}\otimes\rho_{m_j}\circ\phi).$$

Jiang-Soudry's Theorem 2.2 in [17] for $SO(2n+1)$ says

$$L(s,\pi,\rho_{m_i}\otimes\tilde\rho'_{m_0})=L(s,\rho_{m_i}\otimes\tilde\rho'_{m_0}\circ\phi).$$

And the recent work of Henniart [13] tells us

$$L(s,\pi,Sym^2\rho_{m_i})=L(s,Sym^2\rho_{m_i}\circ\phi).$$

Thus,

$$L(s,\pi,r_\beta)=L(s,r_\beta\circ\phi).$$

Therefore, $\mu_\beta(0,\pi)=0$ if and only if $L(s,r_\beta\circ\phi)$ has a pole at $s=0$.

By Lemma 3.2, the Artin L-function $L(s,r_\beta\circ\phi)$ has a pole at $s=0$ if and only if $r_\beta\circ\phi$ contains the one dimensional trivial representation of W'_F. Let $\mathfrak{s}_\phi$ be the Lie algebra of S^o_ϕ. Therefore, $\mu_\beta(0,\pi)=0$ if and only if there is a non-zero vector X_o in ${}^L\mathfrak{g}_{\beta^\vee}$ or ${}^L\mathfrak{g}_{2\beta^\vee}$ such that $(r_\beta\circ\phi(W'_F))(X_o)=X_o$. We can always choose X_o in a neighborhood U of 0 in ${}^L\mathfrak{g}$, where the exponential map exp of ${}^L\mathfrak{g}$ to ${}^LG^o$ exists and is bijective. So $X_o\in\mathfrak{s}_\phi$ if and only if $exp(X_o)\in S^o_\phi$. Since $exp(r_\beta(x)(X))=x^{-1}exp(X)x$ for $x\in{}^LM^o$ and $X\in U$, we see that $\mu_\beta(0,\pi)=0$ if and only if $\mathfrak{s}_\phi\cap{}^L\mathfrak{n}_\beta\neq 0$, and if and only if $\mathfrak{s}_\phi\cap{}^L\mathfrak{g}_{\beta^\vee}\neq 0$ or $\mathfrak{s}_\phi\cap{}^L\mathfrak{g}_{2\beta^\vee}\neq 0$. Therefore, $\beta^\vee$ or $2\beta^\vee$ is in $\Sigma({}^LA^o,S^o_\phi)$, depending on $\mathfrak{s}_\phi\cap{}^L\mathfrak{g}_{\beta^\vee}\neq 0$ or $\mathfrak{s}_\phi\cap{}^L\mathfrak{g}_{2\beta^\vee}\neq 0$. The remark (1) following Lemma 2.1 says that ${}^LA^o$ is a maximal torus of S^o_ϕ, so $\Sigma({}^LA^o,S^o_\phi)$ is a reduced root system. Therefore, not both of $\beta^\vee$ and $2\beta^\vee$ are in $\Sigma({}^LA^o,S^o_\phi)$. Conversely, by reversing the argument above, for each positive root γ in $\Sigma({}^LA^o,S^o_\phi)$, we can associate a unique reduced root $\beta\in\Sigma_r(A,P)$ which is special for π and such that $\beta^\vee$ or $2\beta^\vee$ is equal to γ. $W({}^LA^o,S^o_\phi)$ is generated by the reflections r_γ, where $\gamma\in\Sigma({}^LA^o,S^o_\phi)$. Lemma 2.4 says that $\eta(s_\beta)=s_{\beta^\vee}$. We need to prove that $s_{\beta^\vee}=r_\gamma$, where $\gamma=\beta^\vee$ or $2\beta^\vee$. Let $S_\gamma=Z_{S^o_\phi}(({}^LA^o)_\gamma)$. Since ${}^LA^o$ is a maximal torus of S^o_ϕ, $W({}^LA^o,S_\gamma)$ has order two and $W({}^LA^o,S_\gamma)=\{1,r_\gamma\}$. Note that $S_\gamma\subset({}^LM^o)_{\beta^\vee}$. So, $W({}^LA^o,S_\gamma)\subset W({}^LA^o,({}^LM^o)_{\beta^\vee})=W({}^LA^o,{}^LM^o_\beta)$. Both of them have order two, thus $W({}^LA^o,S_\gamma)=W({}^LA^o,{}^LM^o_\beta)$. It follows $s_{\beta^\vee}=r_\gamma$. Therefore, $W^o(\pi)$ is isomorphic to W^o_ϕ under η. We have finished the proof of (2). □

Remark. The proof of (2) of this theorem can actually be applied to connected reductive split groups, as long as the equalities between Langlands-Shahidi L-functions and corresponding Artin L-functions are established.

4. Generic representations and the Aubert involution

A basic step in computing Langlands parameters for nontempered representations is to find corresponding Langlands data. Lemma 4.2 describes Langlands data for the Aubert involution of a generic representation.

Let us first introduce some notation related to parabolic induction and segments. For admissible representations ρ_1, ρ_2 of $GL(k_1, F)$, $GL(k_2, F)$ respectively, we define

$$\rho_1 \times \rho_2 = Ind_{P(F)}^{G(F)}(\rho_1 \otimes \rho_2),$$

where $P(F)$ is the standard parabolic subgroup of $G(F) = GL(k_1 + k_2, F)$ corresponding to the standard Levi subgroup $M(F) \cong GL(k_1, F) \times GL(k_2, F)$. If ρ is an admissible representation of $GL(k, F)$ and π is an admissible representation of $SO(2m+1, F)$, then we define

$$\rho \times \pi = Ind_{P(F)}^{G(F)}(\rho \otimes \pi),$$

where $P(F)$ is the standard parabolic subgroup of $G(F) = SO(2(k+m)+1, F)$ corresponding to the standard Levi subgroup $M(F) \cong GL(k, F) \times SO(2m+1, F)$ ([32]).

Let ν denote det . Let ρ be an irreducible supercuspidal representation of $GL(k, F)$ and n a non-negative integer. The set $[\rho, \nu^n\rho] = \{\rho, \nu\rho, \ldots, \nu^n\rho\}$ is called a segment. We know from [35] that the representation $\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \rho$ has a unique irreducible subrepresentation $\delta[\rho, \nu^n\rho]$. This subrepresentation is square integrable if the segment is balanced, i.e., of the form $[\nu^{-m}\rho, \nu^m\rho]$, where ρ is unitary and m is half an integer. As in [17], we define

$$\Delta(\rho, m) = \delta[\nu^{-m}\rho, \nu^m\rho],$$

for a balanced segment. The representation $\nu^n\rho \times \nu^{n-1}\rho \times \cdots \times \rho$ has a unique irreducible quotient, which we denote by $\zeta[\rho, \nu^n\rho]$.

For a representation π, we denote by $\tilde{\pi}$ the contragredient of π. If $\tilde{\pi} \cong \pi$, we say that π is self-dual. For a segment $\Sigma = [\rho, \nu^n \rho]$, we define

$$\tilde{\Sigma} = [\nu^{-n}\tilde{\rho}, \tilde{\rho}].$$

Let $D_{M(F)}$ be the Aubert duality operator ([4]). If π is an irreducible representation of $M(F)$, we denote by $\hat{\pi}$ the representation $\pm D_{M(F)}(\pi)$, taking the sign $+$ or $-$ so that $\hat{\pi}$ is a positive element in the Grothendieck group. We call $\hat{\pi}$ the Aubert involution of π. For $GL(n, F)$, the Aubert involution coincides with the Zelevinsky involution and

$$\widehat{\delta(\Sigma)} = \zeta(\Sigma), \quad \widehat{\nu^\alpha \delta(\Sigma)} = \nu^\alpha \zeta(\Sigma). \tag{2}$$

Suppose ρ is an irreducible supercuspidal unitary representation of $GL(k, F)$ and π a generic supercuspidal representation of $SO(2m+1, F)$. If $\rho \not\cong \tilde{\rho}$, then $\nu^\beta \rho \times \pi$ is irreducible, for any $\beta \in \mathbb{R}$. If $\rho \cong \tilde{\rho}$, there exists $\alpha \in \{0, \frac{1}{2}, 1\}$ such that $\nu^{\pm\alpha}\rho \times \pi$ is reducible and $\nu^\beta \rho \times \pi$ is irreducible for $|\beta| \neq \alpha$ [27]. We say the pair (ρ, π) satisfies $(C\alpha)$.

We refer to [24] for the definition of a generic representation, i.e., an irreducible admissible representation having a Whittaker model. We extend this definition to an admissible representation π by saying that π is generic if it has an irreducible generic subquotient. An irreducible admissible representation admits at most one Whittaker model with respect to the generic character θ. A "heredity" property of Whittaker models with respect to parabolic induction is described by Theorem 2 of [24]. We interpret these results for $SO(2n+1, F)$ in the following lemma:

Lemma 4.1. *Let ρ_i, $i = 1, \ldots, q$, be an irreducible admissible representation of $GL(k_i, F)$ and τ an irreducible admissible representation of $SO(2n'+1, F)$. Then the induced representation*

$$\rho_1 \times \cdots \times \rho_q \times \tau$$

has at most one generic component. The representation $\rho_1 \times \cdots \times \rho_q \times \tau$ is generic if and only if ρ_i, $i = 1, \ldots, q$, and τ are generic.

Proof. It follows from [24], Theorems 2 and 3, using the fact that all generic characters of $SO(2n+1, F)$ are T-equivalent. □

Before we proceed, let us review the Langlands classification for $SO(2n+1, F)$. Suppose ρ_i is an irreducible square integrable representation of $GL(n_i, F)$, $i = 1, \ldots, k$ and $\alpha_1 \geq \cdots \geq \alpha_k > 0$ are real numbers. Let τ be a tempered representation of $SO(2\ell+1, F)$. Then the induced representation $\nu^{\alpha_1}\rho_1 \times \cdots \times \nu^{\alpha_k}\rho_k \times \tau$ has a unique irreducible quotient, which we call the Langlands quotient and denote by $L(\nu^{\alpha_1}\rho_1, \ldots, \nu^{\alpha_k}\rho_k; \tau)$. For any irreducible admissible representation π of $SO(2n+1, F)$, there exist Langlands data $\nu^{\alpha_1}\rho_1, \ldots, \nu^{\alpha_k}\rho_k; \tau$ as above, such that $\pi = L(\nu^{\alpha_1}\rho_1, \ldots, \nu^{\alpha_k}\rho_k; \tau)$. Equivalently, we could formulate the Langlands classification with $\rho_1, \ldots, \rho_k$ tempered and $\alpha_1 > \cdots > \alpha_k > 0$, which is the usual form of Langlands data. The connection between two forms of Langlands data comes from the irreducibility of induced-from-unitary-representations of $GL(n, F)$. In particular, if ρ is a tempered representation of $GL(n, F)$, then $\rho \cong \delta_1 \times \cdots \times \delta_s$, for some square integrable representations $\delta_1, \ldots, \delta_s$.

Lemma 4.2. *Let π be an irreducible generic representation of $SO(2n+1, F)$ and $\hat{\pi}$ its Aubert involution. Let*

$$\nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_q}\rho_q \otimes \tau$$

be the Langlands data of $\hat{\pi}$: $\alpha_1 \geq \cdots \geq \alpha_q > 0$, ρ_i is an irreducible square integrable representation of $GL(k_i, F)$, τ is an irreducible tempered representation of $SO(2n'+1, F)$ and $\hat{\pi}$ is the unique quotient of the induced representation

$$\nu^{\alpha_1}\rho_1 \times \cdots \times \nu^{\alpha_q}\rho_q \times \tau.$$

Then ρ_i, $i = 1, \ldots, q$, is supercuspidal and τ is a subrepresentation of

$$\rho_{q+1} \times \cdots \times \rho_r \times \sigma,$$

where ρ_i, $i = q+1, \ldots, r$, is a supercuspidal unitary representation of $GL(k_i, F)$ and σ is a supercuspidal representation of $SO(2n''+1, F)$.

Proof. In the Grothendieck group, the Aubert involution commutes with parabolic induction ([4], Theorem 1.7). By applying the Aubert involution to $\nu^{\alpha_1}\rho_1 \times \cdots \times \nu^{\alpha_q}\rho_q \times \tau$, we conclude that $\pi = \hat{\hat{\pi}}$ is a component of

$$\nu^{\alpha_1}\hat{\rho}_1 \times \cdots \times \nu^{\alpha_q}\hat{\rho}_q \times \hat{\tau}.$$

This representation is generic, because it has the generic subquotient π. Lemma 4.1 tells us that $\hat{\rho}_i$, $i = 1, \ldots, q$, and $\hat{\tau}$ are generic. Let $i \in \{1, \ldots, q\}$. The representation ρ_i is a square integrable representation of $GL(k_i, F)$, therefore, $\rho_i = \delta(\Sigma_i)$, for a balanced segment Σ_i. According to [35], Theorem 9.7, $\hat{\rho}_i = \zeta(\Sigma_i)$ is generic if and only if $\hat{\rho}_i$ is supercuspidal. It follows that ρ_i is supercuspidal.

The tempered representation τ is a subrepresentation of a representation induced from a square integrable representation. Therefore, τ is a subrepresentation of

$$\delta(\Sigma_{q+1}) \times \cdots \times \delta(\Sigma_r) \times \tau_0$$

where the segments $\Sigma_{q+1}, \ldots, \Sigma_r$ are balanced and τ_0 is square integrable. We apply Theorem 1.1 of [15], which describes square integrable representations of odd-orthogonal groups. It follows that τ is a subrepresentation of

$$\delta(\Sigma_{q+1}) \times \cdots \times \delta(\Sigma_r) \times \delta(\Sigma_{r+1}) \times \cdots \times \delta(\Sigma_s) \times \sigma, \tag{3}$$

where the segments $\Sigma_{q+1}, \ldots, \Sigma_r$ are balanced and $\Sigma_{r+1}, \ldots, \Sigma_s$, σ satisfy conditions from Theorem 1.1 of [15]. One of the conditions is that σ is supercuspidal. In the Grothendieck group, the Aubert involution of (3) is equal to

$$\zeta(\Sigma_{q+1}) \times \cdots \times \zeta(\Sigma_r) \times \zeta(\Sigma_{r+1}) \times \cdots \times \zeta(\Sigma_s) \times \sigma. \tag{4}$$

It is generic because it has the generic component $\hat{\tau}$. It follows that $\delta(\Sigma_i)$ is supercuspidal, for all $i \in \{q+1,\dots,r,r+1,\dots,s\}$ and σ is generic. In particular, for $i \in \{q+1,\dots,r\}$, the representation $\delta(\Sigma_i) \cong \rho_i$ is supercuspidal and unitary.

Now, we follow notation of [15]. According to Theorem 1.1 of [15], we consider an irreducible square integrable representation Π, which is a subrepresentation of the representation parabolically induced from $\delta_0(\Pi)$, where

$$\delta_0(\Pi) = \delta[\nu^{-c_1}\rho, \nu^{d_1}\rho] \otimes \cdots \otimes \delta[\nu^{-c_k}\rho, \nu^{d_k}\rho] \otimes \sigma$$

and conditions of Theorem 1.1 [15] are satisfied. We consider the case when $\delta[\nu^{-c_i}\rho, \nu^{d_i}\rho]$ is supercuspidal, for all i, which implies $-c_i = d_i$.

Suppose (ρ, σ) satisfies $(C\alpha)$, i.e., $\alpha \geq 0$ and $\nu^{\pm\alpha}\rho \times \sigma$ is reducible. Since σ is supercuspidal and generic, $\alpha \in \{0, \frac{1}{2}, 1\}$. Let β be as in [15]. Then k is equal to the number of elements in the set $\{-\beta, -\beta-1, \dots, -\alpha\}$. We claim that $k = 0$.

(1) If (ρ, σ) satisfies $(C1)$, then $\alpha = 1$ and $0 < \beta \leq 2$. If $\beta = 1$, then

$$\delta_0(\Pi) = \nu\rho \otimes \sigma.$$

The representation $\nu\rho \times \sigma$ has a unique generic square integrable representation δ. Then $\hat{\delta}$ is not generic. This contradicts the fact that (4) has a generic subquotient. If $\beta = 2$, then $\{-\beta, -\beta-1, \dots, -\alpha\} = \emptyset$.

(2) If (ρ, σ) satisfies $(C\frac{1}{2})$, then $\alpha = \frac{1}{2}$ and $0 < \beta \leq \frac{3}{2}$. The proof is similar to 1.

(3) If (ρ, σ) satisfies $(C0)$, then $\alpha = 0$ and $0 < \beta \leq 1$. It follows that $\beta = 1$ and $\{-\beta, -\beta-1, \dots, -\alpha\} = \emptyset$.

The claim follows. In particular, $\{r+1,\dots,s\} = \emptyset$ and the conditions of the lemma are fulfilled. $\square$

Corollary 4.1. *Let π be an irreducible generic square-integrable representation of $SO(2n+1,F)$ and $\hat{\pi}$ its Aubert involution. Suppose that π is not supercuspidal. Then $\hat{\pi}$ is non-tempered.*

Proof. Follows from Lemma 4.2, using Casselman's square-integrability criterion. □

5. L-parameters and R-groups of generic discrete series and their involutions

We first review the description of all irreducible generic square integrable representations of $SO(2n+1, F)$ in [23] and [33].

Theorem 5.1. *(Muic, [23] and Tadic, [33])*

(a) *Let σ be a generic supercuspidal representation of $SO(2n'+1, F)$ and*

$$\Sigma_i = [\nu^{-a_i}\rho_i, \nu^{b_i}\rho_i],\ 2b_i \in \mathbb{Z}_+,\ 2a_i \in \mathbb{Z},\ \rho_i \cong \tilde{\rho}_i,\ i = 1, \ldots, k$$

a set of segments satisfying

(i) $b_i > a_i$.

(ii) (1) *If (ρ_i, σ) satisfies $(C\frac{1}{2})$, then $b_i \in \frac{1}{2} + \mathbb{Z}$, $a_i \geq -\frac{1}{2}$.*

(2) *If (ρ_i, σ) satisfies $(C0)$, then $b_i \in \mathbb{Z}$, $a_i \geq 0$.*

(3) *If (ρ_i, σ) satisfies $(C1)$, then $b_i \in \mathbb{Z}$, $a_i \geq -1$, $a_i \neq 0$.*

(iii) *If $\rho_i \cong \rho_j$ for $i \neq j$, then either $b_i < a_j$ or $b_j < a_i$.*

Then, the representation $\delta(\Sigma_1 \cap \tilde{\Sigma}_1) \times \cdots \times \delta(\Sigma_k \cap \tilde{\Sigma}_k) \times \sigma$ has a unique irreducible generic subrepresentation, denote it by τ. The representation $\delta(\Sigma_1 \setminus \tilde{\Sigma}_1) \times \cdots \times \delta(\Sigma_k \setminus \tilde{\Sigma}_k) \times \tau$ has a unique irreducible subrepresentation which we denote by

$$\delta(\Sigma_1, \ldots, \Sigma_k, \sigma)_\tau.$$

The representation $\delta(\Sigma_1, \ldots, \Sigma_k, \sigma)_\tau$ is square integrable, generic.

(b) *Suppose π is an irreducible square integrable generic representation of $SO(2n+1, F)$. Then there exists a unique σ and a unique set of segments $\{\Sigma_1, \ldots, \Sigma_k\}$ satisfying (i) - (iii) such that $\pi \cong \delta(\Sigma_1, \ldots, \Sigma_k, \sigma)_\tau$.*

We fix $\pi \cong \delta(\Sigma_1, \ldots, \Sigma_k, \sigma)_\tau$ as in Theorem 5.1. Let

$$P = \{1, \ldots, k\}.$$

Denote by r the local Langlands reciprocity map for $GL(F)$ [11, 12]. Let ϕ_i be the Langlands parameter for ρ_i, i.e., $r(\phi_i) = \rho_i$. For $\alpha = 0, \frac{1}{2}, 1$, define the following subsets of P:

$$C_\alpha = \{i \in P \mid (\rho_i, \sigma) \text{ satisfies } (C\alpha), a_i \geq 0\},$$

$$C_{-1} = \{i \in P \mid a_i = -1\},$$

$$C_{-\frac{1}{2}} = \{i \in P \mid a_i = -\frac{1}{2}\},$$

$$P_0 = \{i \in P \mid a_i \geq 0\} = C_0 \cup C_{\frac{1}{2}} \cup C_1.$$

Lemma 5.1. *Let $\pi = \delta(\Sigma_1, \ldots, \Sigma_k, \sigma)_\tau$. Let $\{l_1, \ldots, l_t\}$ be the multiset*

$$\{ j_i \mid i \in P, j_i \in \{-a_i, -a_i + 1, \ldots, b_i\} \setminus \{0\}\}$$

written in a non-increasing order. For $l_s = j_i$, let $\rho_{l_s} = \rho_i$. Let τ_0 be the unique generic component of

$$\left(\underset{i \in C_0 \cup C_1}{\times} \rho_i \right) \times \sigma.$$

Then $\hat{\pi}$ is the Langlands quotient of the induced representation

$$\nu^{l_1} \rho_{l_1} \times \cdots \times \nu^{l_t} \rho_{l_t} \times \hat{\tau}_0.$$

Proof. Let

$$\Pi = (\nu^{b_1} \rho_1 \otimes \nu^{b_1 - 1} \rho_1 \otimes \cdots \otimes \nu^{-a_1} \rho_1) \otimes \cdots \otimes (\nu^{b_k} \rho_k \otimes \nu^{b_k - 1} \rho_k \otimes \cdots \otimes \nu^{-a_k} \rho_k) \otimes \sigma.$$

Denote by $P(F)$ the standard parabolic subgroup corresponding to Π. We consider the full-induced representation $Ind_{P(F)}^{G(F)} \Pi$. Then π is a subrepresentation of $Ind_{P(F)}^{G(F)} \Pi$ and, by Corollary 4.2 of [5], $\hat{\pi}$ is a quotient of $Ind_{P(F)}^{G(F)} \Pi$. Lemma 4.2 tells us that $\hat{\pi}$ is a quotient of

$$\nu^{\alpha_1} \delta_1 \times \cdots \times \nu^{\alpha_q} \delta_q \times \delta_{q+1} \times \cdots \times \delta_r \times \sigma,$$

where $\alpha_1 \geq \cdots \geq \alpha_q > 0$ and δ_i are supercuspidal unitary representations. It follows from [9], Corollary 6.3.7 and from the description of the Weyl group for odd-orthogonal groups ([32]) that $\nu^{\alpha_1}\delta_1 \otimes \cdots \times \nu^{\alpha_q}\delta_q \otimes \delta_{q+1} \otimes \cdots \otimes \delta_r \otimes \sigma$ can be obtained from Π by permutations and taking contragredients. The condition on $\alpha_1, \ldots, \alpha_q$ implies that $\{\alpha_1, \ldots, \alpha_q\} = \{l_1, \ldots, l_t\}$. Therefore, $\hat{\pi}$ is a quotient of

$$\nu^{l_1}\rho_{l_1} \times \cdots \times \nu^{l_t}\rho_{l_t} \times \left(\bigtimes_{i \in C_0 \cup C_1} \rho_i \right) \times \sigma.$$

In particular, $\hat{\pi}$ is a quotient of the induced representation

$$\nu^{l_1}\rho_{l_1} \times \cdots \times \nu^{l_t}\rho_{l_t} \times \tau', \tag{5}$$

where τ' is a component of

$$\left(\bigtimes_{i \in C_0 \cup C_1} \rho_i \right) \times \sigma.$$

Note that (5) is Langlands data. It has the unique quotient $\hat{\pi}$. Then π is a component of $(\nu^{l_1}\rho_{l_1} \times \cdots \times \nu^{l_t}\rho_{l_t} \times \tau')\hat{}$, which has the same irreducible factors as $\nu^{l_1}\rho_{l_1} \times \cdots \times \nu^{l_t}\rho_{l_t} \times \hat{\tau}'$. We conclude that $\hat{\tau}'$ is generic, so $\tau' = \hat{\tau}_0$. $\square$

Theorem 5.2. *Let π be an irreducible generic square integrable representation of $SO(2n+1, F)$. Write $\pi \cong \delta(\Sigma_1, \ldots, \Sigma_k, \sigma)_\tau$ as in Theorem 5.1. Let*

$$\ell(\pi) = \rho_{k+1} \times \cdots \times \rho_l$$

be the local Langlands functorial lift of π defined in [17]. Define

$$A = \{k+1, \ldots, l\},$$

$$A_0 = A \setminus \{j \in A \mid \rho_j \cong \rho_i \text{ for some } i \in C_{-1}\}.$$

Then

$$\ell(\pi) = \left(\bigtimes_{i \in P} \Delta(\rho_i, b_i) \right) \times \left(\bigtimes_{i \in P_0} \Delta(\rho_i, a_i) \right) \times \left(\bigtimes_{i \in A_0} \rho_i \right).$$

The Langlands parameter for π is

$$\left(\bigoplus_{i \in P} \phi_i \otimes S_{2b_i+1} \right) \oplus \left(\bigoplus_{i \in P_0} \phi_i \otimes S_{2a_i+1} \right) \oplus \left(\bigoplus_{i \in A_0} \phi_i \otimes S_1 \right). \tag{6}$$

The Langlands parameter for $\hat{\pi}$ is

$$\left(\bigoplus_{i \in P} \bigoplus_{\substack{j=-a_i \\ j \neq 0}}^{b_i} \left(\cdot \; {}^{|j} \phi_i \otimes S_1 \oplus \; \cdot \; {}^{-|j} \phi_i \otimes S_1 \right) \right)$$

$$\oplus \left(\bigoplus_{i \in C_0 \cup C_1} \phi_i \otimes S_1 \right) \oplus \left(\bigoplus_{i \in A} \phi_i \otimes S_1 \right) \oplus \left(\bigoplus_{i \in C_0 \cup C_1} \phi_i \otimes S_1 \right),$$

which is equal to

$$\left(\bigoplus_{i \in P} \bigoplus_{j=-a_i}^{b_i} \left(\cdot \; {}^{|j|} \phi_i \otimes S_1 \oplus \; \cdot \; {}^{-|j|} \phi_i \otimes S_1 \right) \right) \oplus \left(\bigoplus_{i \in A} \phi_i \otimes S_1 \right). \tag{7}$$

Proof. According to [16], Theorem 6.1 and [17], the proof of Theorem 2.1 (also, cf. [36], Proposition 4.1), we have

(i) If $i, j \in A$, $i \neq j$, then $\rho_i \not\cong \rho_j$.
(ii) $\{\rho_i \;\; i \in A\} = \{\rho \;\; (\rho, \sigma) \, satisfies (C1)\}$.

The local Langlands functorial lift of σ follows from [17], the proof of Theorem 2.1. According to [17], equation (2.32), the lift $\ell(\pi)$ is the generic constituent of

$$\times_{i \in P} \left(\delta(\Sigma_i) \times \delta(\tilde{\Sigma}_i) \right) \times \ell(\sigma). \tag{8}$$

The proof of Theorem 2.1 in [17] describes different pieces of (8) and corresponding generic constituents. The P_0-piece comes from equations (2.39), (2.43) and (2.45) of [17]:

$$\times_{i \in P_0} (\Delta(\rho_i, a_i) \times \Delta(\rho_i, b_i)). \tag{9}$$

We obtain the C_{-1}-piece from equation (2.41) of [17], by eliminating the part which is included in (9). The C_{-1}-piece is

$$\times_{i \in C_{-1}} \Delta(\rho_i, b_i). \tag{10}$$

In a similar way, we obtain the $C_{-\frac{1}{2}}$-piece from equation (2.47) of [17]:

$$\times_{i \in C_{-\frac{1}{2}}} \Delta(\rho_i, b_i). \tag{11}$$

Now the lift $\ell(\pi)$ follows from (9), (10) and (11):

$$\ell(\pi) = \left(\times_{i\in P} \Delta(\rho_i, b_i) \right) \times \left(\times_{i\in P_0} \Delta(\rho_i, a_i) \right) \times \left(\times_{i\in A_0} \rho_i \right).$$

Let C_0^* be a subset of C_0 such that $\{\rho_i \mid i \in C_0^*\}$ contains exactly one copy of ρ, for each $\rho \in \{\rho_i \mid i \in C_0\}$. Let $C_0^{**} = C_0 \setminus C_0^*$. Denote by τ_0 the unique generic constituent of

$$\left(\times_{i\in C_0\cup C_1} \rho_i \right) \times \sigma = \left(\times_{i\in C_1\cup C_0^{**}} \rho_i \right) \times \left(\times_{i\in C_0^*} \rho_i \right) \times \sigma. \tag{12}$$

The representation $\left(\times_{i\in C_0^*} \rho_i \right) \times \sigma$ has a unique generic subrepresentation τ_1, which is elliptic. Then by [17], the proof of Theorem 3.1, equation (3.12),

$$\ell(\tau_1) = \left(\times_{i\in C_0^*} \rho_i \right) \times \left(\times_{i\in A} \rho_i \right) \times \left(\times_{i\in C_0^*} \rho_i \right).$$

The representation $\left(\times_{i\in C_1\cup C_0^{**}} \rho_i \right) \times \tau_1$ is irreducible, generic and it is equal to τ_0. According to [17], the proof of Theorem 4.1, equation (4.25),

$$\ell(\tau_0) = \left(\times_{i\in C_1\cup C_0^{**}} \rho_i \right) \times \ell(\tau_1) \times \left(\times_{i\in C_1\cup C_0^{**}} \rho_i \right) = \left(\times_{i\in C_0\cup C_1} \rho_i \right) \times \left(\times_{i\in A} \rho_i \right) \times \left(\times_{i\in C_0\cup C_1} \rho_i \right).$$

The parameter for τ_0 is

$$\left(\bigoplus_{i\in C_0\cup C_1} \phi_i \otimes S_1 \right) \oplus \left(\bigoplus_{i\in A} \phi_i \otimes S_1 \right) \oplus \left(\bigoplus_{i\in C_0\cup C_1} \phi_i \otimes S_1 \right).$$

Let $\{l_1, \ldots, l_t\}$ be the multiset

$$\{\, j_i \mid i \in P, \, j_i \in \{-a_i, -a_i+1, \ldots, b_i\} \setminus \{0\}\}$$

written in a non-increasing order. Then $\hat{\pi}$ is the Langlands quotient of the induced representation $\nu^{l_1}\rho_{l_1} \times \cdots \times \nu^{l_t}\rho_{l_t} \times \hat{\tau}_0$. The Langlands parameter of a Langlands quotient (for $SO(2n+1,F)$) is described in [17], Proposition 6.1 and Theorem 6.1. According to equations (6.5), (6.9) and (6.2) of [17], the Langlands parameter of $\hat{\pi}$ is

$$\left(\bigoplus_{s=1}^{t}\left(\cdot\ {}^{l_s} r^{-1}(\rho_{l_s}) \otimes S_1 \oplus \ \cdot\ {}^{-l_s} r^{-1}(\rho_{l_s}) \otimes S_1\right)\right) \oplus \varphi(\hat{\tau}_0)$$

$$= \left(\bigoplus_{i\in P}\bigoplus_{\substack{j=-a_i\\ j\neq 0}}^{b_i}\left(\cdot\ |^{j}\, \phi_i \otimes S_1 \oplus \ \cdot\ {}^{-j}\, \phi_i \otimes S_1\right)\right) \oplus \varphi(\hat{\tau}_0),$$

which is equal to

$$\left(\bigoplus_{i\in P}\bigoplus_{\substack{j=-a_i\\ j\neq 0}}^{b_i}\left(\cdot\ |^{j}\, \phi_i \otimes S_1 \oplus \ \cdot\ {}^{-|j}\, \phi_i \otimes S_1\right)\right)$$

$$\oplus\left(\bigoplus_{i\in C_0\cup C_1}\phi_i\otimes S_1\right)\oplus\left(\bigoplus_{i\in A}\phi_i\otimes S_1\right)\oplus\left(\bigoplus_{i\in C_0\cup C_1}\phi_i\otimes S_1\right).$$

Here, we use the fact that τ_0 and $\hat{\tau}_0$ have the same parameter, which can be explained as follows. From (12), τ_0 is the unique generic constituent of

$$\left(\times_{i\in C_0\cup C_1}\rho_i\right)\times\sigma. \tag{13}$$

Corollary 4.2 of [5] tells us $\hat{\tau}_0$ is a component of (13). Therefore, τ_0 and $\hat{\tau}_0$ are tempered representations induced from the same discrete series representation. Langlands' original construction of L-packets for real groups, from discrete series L-packets, based on the Langlands classification [21], can be repeated for p-adic groups. Tempered L-packets are then defined by inducing from discrete series L-packets on Levi factors ([27], Section 9). In particular, tempered representations coming from a single discrete series are a part of the same L-packet. $\square$

Corollary 5.1. *Let π be a generic discrete representation of $G(F)$. Let*

$$\psi: W_F\times SL(2,\mathbb{C})\times SL(2,\mathbb{C})\to Sp(2n,\mathbb{C})$$

be the A-parameter of π and $\hat{\psi}$ the A-parameter of $\hat{\pi}$. Then,

$$\hat{\psi}(w, x, y) = \psi(w, y, x).$$

In particular, $\hat{\psi}$ and ψ have the same image in $Sp(2n, \mathbb{C})$.

Proof. First, let us consider an A-parameter of the form

$$\psi_1 = \phi \otimes S_m \otimes S_n. \tag{14}$$

Let ϕ_{ψ_1} denote the L-parameter corresponding to ψ_1, defined by equation (1). From the definition of S_n, which is the n dimensional irreducible complex representation of $SL_2(\mathbb{C})$, we see that

$$\phi(w) \otimes S_n(\begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}) = \bigoplus_{-(n-1)/2 \le j \le (n-1)/2} \phi(w)\, |w|^{j}.$$

It follows

$$\phi_{\psi_1} = \bigoplus_{j=-(n-1)/2}^{(n-1)/2} |\cdot|^{j} \phi \otimes S_m. \tag{15}$$

Now, let $\Sigma_1, \cdots, \Sigma_k, \sigma$ and $\pi = \delta(\Sigma_1, \cdots, \Sigma_k, \sigma)_\tau$ be as in Theorem 5.2. The L-parameters of π and $\hat{\pi}$ are computed in Theorem 5.2. Denote by ψ the A-parameter of π. Since π is tempered, we read ψ directly from (6):

$$\psi = \left(\bigoplus_{i \in P} \phi_i \otimes S_{2b_i+1} \otimes S_1\right) \oplus \left(\bigoplus_{i \in P_0} \phi_i \otimes S_{2a_i+1} \otimes S_1\right) \oplus \left(\bigoplus_{i \in A_0} \phi_i \otimes S_1 \otimes S_1\right). \tag{16}$$

Define

$$\hat{\psi} = \left(\bigoplus_{i \in P} \phi_i \otimes S_1 \otimes S_{2b_i+1}\right) \oplus \left(\bigoplus_{i \in P_0} \phi_i \otimes S_1 \otimes S_{2a_i+1}\right) \oplus \left(\bigoplus_{i \in A_0} \phi_i \otimes S_1 \otimes S_1\right). \tag{17}$$

We claim $\hat{\psi}$ is the A-parameter of $\hat{\pi}$. Indeed, (15) and (17) imply

$$\phi_{\hat{\psi}} = \left(\bigoplus_{i \in P} \bigoplus_{j=-b_i}^{b_i} |\cdot|^{j} \phi_i \otimes S_1\right) \oplus \left(\bigoplus_{i \in P_0} \bigoplus_{j=-a_i}^{a_i} |\cdot|^{j} \phi_i \otimes S_1\right) \oplus \left(\bigoplus_{i \in A_0} \phi_i \otimes S_1\right)$$

$$= \left(\bigoplus_{i \in P} \bigoplus_{j=-a_i}^{b_i} (|\cdot|^{j} \phi_i \otimes S_1 \oplus |\cdot|^{-j} \phi_i \otimes S_1) \right) \oplus \left(\bigoplus_{i \in A} \phi_i \otimes S_1 \right).$$

This is precisely the L-parameter of $\hat{\pi}$, given by (7). Therefore, $\hat{\psi}$ is the A-parameter of $\hat{\pi}$. If we compare (16) and (17), we see that $\hat{\psi}(w, x, y) = \psi(w, y, x)$. It follows that ψ and $\hat{\psi}$ have the same image in ${}^L G$. □

Theorem 5.3. *Let $G = SO(2n+1)$, $M = GL_{m_1} \times \cdots \times GL_{m_r} \times G_{m_0}$ a Levi subgroup of G, and $\pi = \pi_1 \otimes \cdots \otimes \pi_r \otimes \pi_0$ a generic discrete representation of $M(F)$. Assume that all the members in the L-packet containing π have same Plancherel measures. In addition, assume $\hat{\pi}$ is unitary. Then*

$$R_{\psi, \hat{\pi}} \simeq R(\hat{\pi}), \quad \textit{where } \psi = \psi(\hat{\pi}).$$

Proof.

$$\psi(\hat{\pi}) = \psi(\hat{\pi}_1) \oplus \cdots \oplus \psi(\hat{\pi}_r) \oplus \psi(\hat{\pi}_0).$$

By Corollary 5.1, $\psi(\hat{\pi}_0)$ and $\phi(\pi_0)$ have the same image. Let $i \in \{1, \ldots, r\}$. Since π_i is square integrable, it is of the form $\pi_i = \Delta(\rho_i, \frac{b_i - 1}{2})$, where $b_i \in \mathbb{Z}$ and ρ_i is a unitary supercuspidal representation of $GL_{b_i}(F)$. Then $\phi(\pi_i) = \phi(\rho_i) \otimes S_{b_i}$ ([35], Section 10). It follows from (2) that $\hat{\pi}_i$ is the Langlands quotient of the induced representation $\nu^{(b_i-1)/2} \rho_i \times \cdots \times \nu^{-(b_i-1)/2} \rho_i$. According to [35], Section 10, the L-parameter of $\hat{\pi}_i$ is

$$\phi(\hat{\pi}_i) = |\cdot|^{(b_i-1)/2} \phi(\rho_i) \times \cdots \times |\cdot|^{-(b_i-1)/2} \phi(\rho_i).$$

Equations (14) and (15) imply

$$\psi(\hat{\pi}_i) = \phi(\rho_i) \otimes S_{b_i} \otimes S_1.$$

So $\psi(\hat{\pi}_i)$ and $\phi(\pi_i)$ have the same image. Therefore, $\psi(\hat{\pi})$ and $\phi(\pi)$ have the same image in ${}^L G$. Let $\psi = \psi(\hat{\pi})$, $\phi = \phi(\pi)$. It follows that

$$W_\psi = W_\phi, \quad W_\psi^o = W_\phi^o.$$

$\hat{\pi}$ is certainly an element in the A-packet $\Pi_\psi(M)$ of ψ [1]. By definition,

$$W_{\psi,\hat{\pi}} = \{\omega \in W_\psi;\ \ \omega\hat{\pi} \simeq \hat{\pi}\},$$

$$W^o_{\psi,\hat{\pi}} = \{\omega \in W^o_\psi;\ \ \omega\hat{\pi} \simeq \hat{\pi}\}.$$

Theorem 3.1 says that $W_\phi \simeq W(\pi)$ and $W^o_\phi \simeq W^o(\pi)$. Since for $\omega \in W(A, G)$, $\omega\pi \simeq \pi$ if and only if $\omega\hat{\pi} \simeq \hat{\pi}$, we see

$$W_{\psi,\hat{\pi}} \simeq W(\pi),\ \ W^o_{\psi,\hat{\pi}} \simeq W^o(\pi).$$

This gives $R_{\psi,\hat{\pi}} \simeq R(\pi)$. The classical R-group $R(\hat{\pi})$ of $\hat{\pi}$ is defined in [5] by putting $R(\hat{\pi}) = R(\pi)$. So $R_{\psi,\hat{\pi}} \simeq R(\hat{\pi})$. $\square$

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