THE LANGLANDS CLASSIFICATION FOR NON-CONNECTED p-ADIC GROUPS II: MULTIPLICITY ONE

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Abstract. For a non-connected reductive p-adic group, we prove that the Langlands subrepresentation appears with multiplicity one in the representation parabolically induced from the corresponding Langlands data.

1. Introduction

The Langlands classification for a non-connected reductive p-adic group G [B-J1] gives a bijective correspondence

$$Irr(G) \longleftrightarrow Lang(G)$$

between irreducible, admissible representations of G and triples of Langlands data. Let (P, ν, τ) be a set of Langlands data (see Definition 2.1) and suppose that π is the irreducible representation of G corresponding to (P, ν, τ),

$$\pi \longleftrightarrow (P, \nu, \tau).$$

Then π is the unique irreducible subrepresentation of the induced representation $i_{G,M}(exp\, \nu \otimes \tau)$. This paper proves that the multiplicity of π in $i_{G,M}(exp\, \nu \otimes \tau)$ is one (cf. Theorem 3.4). Our motivation goes beyond a general interest in extending some useful properties of the Langlands classification to the non-connected case; we have need of them in other work (cf. [B-J2],[J]).

Before closing the introduction, we would like to take the opportunity to thank M. Tadić for conversations helpful to this work.

2. The Langlands classification

We take a moment to review the Langlands classification in the context of non-connected p-adic groups (cf. [B-J1]; also see [B-W] and [S] for connected p-adic groups, [L] for connected real groups, and [M] for non-connected real groups).

Let F be a p-adic field and G be the group of F-points of a quasi-split reductive algebraic group defined over F. Let G^0 denote the connected component of the identity in G. Assume that

$$C = G/G^0$$

is a finite abelian group.

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We call an irreducible representation of G tempered if its restriction to G^0 is tempered (cf. Definition 2.5 [B-J1]).

In the group G^0, fix a Borel subgroup $P_\emptyset \subset G^0$ and a maximal split torus $A_\emptyset \subset P_\emptyset$. We let Π denote the corresponding set of simple roots. We can choose a set of representatives for C which stabilize $P_\emptyset$, hence act on Π. By abuse of notation, we use C for both the component group and this set of representatives.

For $\Phi \subset \Pi$, we let $P_\Phi = M_\Phi U_\Phi$ denote the standard parabolic subgroup of G^0 determined by Φ. Fix an order on Π. Then, there is a lexicographic order on subsets of Π. We define

$$X_C = \{\Phi \subset \Pi \,|\, \Phi \text{ is maximal among } \{c \cdot \Phi\}_{c \in C}\}.$$

Let $C(\Phi) = \{c \in C \,|\, c \cdot \Phi = \Phi\}$ and

$$M_{\Phi, C(\Phi)} = \langle M_\Phi, C(\Phi) \rangle.$$

We call $P = MU_\Phi$, where $M_\Phi \leq M \leq M_{\Phi, C(\Phi)}$ and $\Phi \in X_C$, a standard parabolic subgroup of G.

Suppose P is a standard parabolic subgroup of G. Write $P^0 = P_\Phi$. Let A be the split component of M_Φ, $\mathfrak{a}$ the real Lie algebra of A, and $\mathfrak{a}^*$ its dual. Let $\Pi(P^0, A) \subset \mathfrak{a}^*$ denote the set of simple roots corresponding to the pair (P^0, A). We set

$$\begin{aligned} \mathfrak{a}^*_- &= \{x \in \mathfrak{a}^* \,|\, \langle x, \alpha \rangle < 0, \, \forall \alpha \in \Pi(P^0, A)\}, \\ \mathfrak{a}^*_-(C) &= \{x \in \mathfrak{a}^*_- \,|\, x \succeq c \cdot x, \, \forall c \in C(\Phi)\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is a $C(\Phi)$-invariant inner product on $\mathfrak{a}^* \times \mathfrak{a}^*$ and $\succeq$ is the lexicographic order inherited from the order on Π (cf. section 3 [B-J1] for details).

Definition 2.1. ([B-J1] Definition 4.1) *A* **set of Langlands data** *for G is a triple (P, ν, τ) with the following properties:*

1. *$P = MU$ is a standard parabolic subgroup of G.*
2. *$\nu \in \mathfrak{a}^*_-(C)$.*
3. *$M = M_{\Phi, C(\Phi,\nu)}$, where $C(\Phi, \nu) = \{c \in C(\Phi) \,|\, c \cdot \nu = \nu\}$.*
4. *$\tau \in Irr(M)$ is tempered.*

For $\nu \in \mathfrak{a}^*_-(C)$, let $exp\,\nu$ be the character of M_Φ defined by $exp\,\nu = q^{\langle \nu, H_\Phi(\cdot) \rangle}$, where $H_\Phi : M_\Phi \to \mathfrak{a}$ is the homomorphism defined in [H] . If τ is a representation of M, then $exp\,\nu \otimes \tau$ is the representation of M defined by $(exp\,\nu \otimes \tau)(mc) = exp\,\nu(m)\tau(mc)$ ([B-J1] section 2). As in [B-Z], we let $i_{G,M}(\tau)$ denote the representation of G parabolically induced from τ.

Theorem 2.2. (Langlands classification, [B-J1] Theorem 4.2)
There is a bijective correspondence

$$Lang(G) \longleftrightarrow Irr(G),$$

where Lang(G) denotes the set of all triples of Langlands data. Further, if $(P, \nu, \tau) \leftrightarrow \pi$ *under this correspondence, then* π *is the unique irreducible subrepresentation of* $i_{G,M}(exp\,\nu \otimes \tau)$.

If $(P, \nu, \tau) \leftrightarrow \pi$, then we call π the Langlands subrepresentation corresponding to (P, ν, τ).

3. Multiplicity one

We now take up the proof of multiplicity one for the Langlands classification.

In what follows, it will occasionally be convenient to work in the Grothendieck group setting. Recall that in this context, we write $\pi_1 \leq \pi_2$ if $m(\tau, \pi_1) \leq m(\tau, \pi_2)$ for every smooth, irreducible representation τ of G, where $m(\tau, \pi)$ =multiplicity of τ in π.

Let $\mathfrak{a}_0$ denote the real Lie algebra of $A_\emptyset$ and $\mathfrak{a}_0^*$ its dual. Recall that for a standard parabolic subgroup, we may identify $\mathfrak{a}^*$ as a subspace of $\mathfrak{a}_0^*$. Let $>$ be the order from section XI.2 [B-W], i.e.,

$$\mu > \nu \text{ if } \langle \mu - \nu, \alpha \rangle > 0, \text{ for all } \alpha \in \Pi,$$

or equivalently, $\mu - \nu \in (\mathfrak{a}_0^*)_+$. We write $\mu \geq \nu$ if $\langle \mu - \nu, \alpha \rangle \geq 0$, for all $\alpha \in \Pi$. Note that $\mu \geq \nu$ and $\nu \geq \mu$ imply that $\mu - \nu$ is in the center of $\mathfrak{a}_0^*$ (e.g., see [B-W], XI.1).

Lemma 3.1. *Let* $\mu, \nu \in \mathfrak{a}_0^*$.

1. *If* $\mu > \nu$, *then* $c \cdot \mu > c \cdot \nu$, *for all* $c \in C$.
2. *For all* $c \in C$, $\mu \not> c \cdot \mu$ *and* $\mu \not< c \cdot \mu$.

Proof. 1. This follows immediately from the fact that $c \cdot \Pi = \Pi$ and the fact that the inner product on $\mathfrak{a}_0^* \times \mathfrak{a}_0^*$ may be taken to be C-invariant (cf. section 2 [B-J1]).

2. Suppose $\mu > c \cdot \mu$, for some $c \in C$. Let m be the order of c. Then, according to 1., $\mu > c \cdot \mu$ implies $c \cdot \mu > c^2 \cdot \mu$, $c^2 \cdot \mu > c^3 \cdot \mu$, etc. Thus,

$$\mu > c \cdot \mu > c^2 \cdot \mu > \cdots > c^m \cdot \mu = \mu,$$

a contradiction. □

We now define an ordering on the C-orbits in $\mathfrak{a}_0^*$ and show it is well-defined.

Definition 3.2. *Let* $\mu, \nu \in \mathfrak{a}_0^*$. *We write* $C \cdot \mu > C \cdot \nu$ *if* $c_1 \cdot \mu > c_2 \cdot \nu$, *for some* $c_1, c_2 \in C$.

Lemma 3.3. *The ordering in Definition 3.2 is well-defined. In particular, suppose* $\mu, \nu \in \mathfrak{a}_0^*$. *Then, the folowing hold:*

1. *If* $C \cdot \mu > C \cdot \nu$, *then* $C \cdot \nu \not> C \cdot \mu$.
2. $C \cdot \mu \not> C \cdot \mu$.

Proof. 1. Suppose $C \cdot \mu > C \cdot \nu$ and $C \cdot \nu > C \cdot \mu$. According to Lemma 3.1, there exist $c_1, c_2 \in C$ such that $\mu > c_1 \cdot \nu$ and $\nu > c_2 \cdot \mu$. Then, by Lemma 3.1, 1.,

$$\mu > c_1 \cdot \nu > c_1 c_2 \cdot \mu,$$

which is a contradiction (Lemma 3.1, 2).

2. Follows from Lemma 3.1, 2. □

Theorem 3.4. *Let π be an irreducible representation of G having Langlands data (P, ν, τ). Then,*

1. *π appears with multiplicity one in $i_{G,M}(exp\, \nu \otimes \tau)$.*
2. *Suppose that θ is an irreducible component of $i_{G,M}(exp\, \nu \otimes \tau)$ with Langlands data $(P_\theta, \nu_\theta, \tau_\theta)$. Then, $C \cdot \nu < C \cdot \nu_\theta$ or $\nu = \nu_\theta$. Further, equality occurs if and only if $\theta \cong \pi$.*

Proof. Let

$$G^0 = G_0 \subset G_1 \subset \cdots \subset G_k = G,$$

where $|G_i/G_{i-1}|$ is prime for $i = 1, \ldots, k$. Recall that the proof of the Langlands classification in [B-J1] follows the lead of [G-H], using a result of [G-K] (cf. Lemma 2.1 [B-J1] for a precise formulation) to work inductively up the filtration. Thus, as in [B-J1], we argue inductively, assuming that multiplicity one holds for G_{i-1} and showing that it holds for G_i. Multiplicity one for the connected group G^0 is classical result (e.g., see [B-W]).

For convenience, let $G_1 \subset G_2$ be two consecutive groups in the filtration above (not necessarily the first two). Then $G_1/G^0 = C_1$ and $G_2/G^0 = C_2$ with $C_1 \subset C_2 \subset C$ and $|C_2/C_1|$ prime. Write $D = C_2/C_1$.

Let π_1 be an irreducible representation of G_1 with Langlands data (P_1, ν_1, τ_1). Write $P_1^0 = P_{\Phi_1}$. For $i = 1, 2$, let

$$\begin{aligned} C_i(\Phi_1) &= \{c \in C_i \mid c \cdot \Phi_1 = \Phi_1\}, \\ C_i(\Phi_1, \nu_1) &= \{c \in C_i \mid c \cdot \Phi_1 = \Phi_1 \text{ and } c \cdot \nu_1 = \nu_1\}, \\ C_i(\Phi_1, \nu_1, \tau_1) &= \{c \in C_i \mid c \cdot \Phi_1 = \Phi_1, c \cdot \nu_1 = \nu_1 \text{ and } c \cdot \tau_1 = \tau_1\}. \end{aligned}$$

According to [B-J1], Lemma 4.3, we have either $C_2(\cdot) = C_1(\cdot)$ or $C_2(\cdot)/C_1(\cdot) \cong D$.

Lemma 3.5. *Assume Theorem 3.4 holds for G_1. Let π_2 be an irreducible representation of G_2 having Langlands data (P_2, ν_2, τ_2). Then, π_2 appears with multiplicity one in $i_{G_2,M_2}(exp\, \nu_2 \otimes \tau_2)$.*

Proof. Let π_1 be an irreducible subquotient of $r_{G_1,G_2}(\pi_2)$. We denote its Langlands data by (P_1, ν_1, τ_1). Recall that the proof of the Langlands classification in [B-J1] considers four cases:

1. $C_2(\Phi_1) = C_1(\Phi_1)$.
2. $C_2(\Phi_1) \neq C_1(\Phi_1)$ but $C_2(\Phi_1, \nu_1) = C_1(\Phi_1, \nu_1)$.

3. $C_2(\Phi)\neq C_1(\Phi)$, $C_2(\Phi,\nu_1)\neq C_1(\Phi,\nu_1)$ but $C_2(\Phi,\nu_1,\tau_1)=C_1(\Phi,\nu_1,\tau_1)$.
4. $C_2(\Phi)\neq C_1(\Phi)$, $C_2(\Phi,\nu_1)\neq C_1(\Phi,\nu_1)$, $C_2(\Phi,\nu_1,\tau_1)\neq C_1(\Phi,\nu_1,\tau_1)$.

We claim the following: If $c\in C_2$ and $c\cdot\pi_1\not\cong\pi_1$, then $c\cdot\pi_1$ is not a subquotient of $i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)$. If (P_1,ν_1,τ_1) fits into case 1 above, then $c\cdot\pi_1$ has Langlands data $(c\cdot P_1,c\cdot\nu_1,c\cdot\tau_1)$ (cf. Proposition 4.5 [B-J1]). However, if θ_1 is an irreducible subquotient of $i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)$ having Langlands data $(P_{\theta_1},\nu_{\theta_1},\tau_{\theta_1})$, then Theorem 3.4 for G_1 implies $C_1\cdot\nu_{\theta_1}>C_1\cdot\nu_1$ or $\nu_{\theta_1}=\nu_1$, $\theta_1\cong\pi_1$. Since this is not the case for $c\cdot\pi_1$ (cf. Lemma 3.1 2.), it follows that $c\cdot\pi_1$ is not a subquotient of $i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)$. The argument when (P_1,ν_1,τ_1) is case 2 is essentially identical. The only remaining possibility is case 3 (case 4 has $c\cdot\pi_1\cong\pi_1$, for all $c\in C_2$). We note that in this case $\nu_{c\cdot\pi_1}=c\cdot\nu_1=\nu_1$. Since π_1 is the only irreducible subquotient of $i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)$ having exponent ν_1 (Theorem 3.4 for G_1), we see that $c\cdot\pi_1$ is not a subquotient of $i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)$, finishing the claim.

We first consider the cases which have $i_{G_2,G_1}(\pi_1)\cong\pi_2$, that is, when (P_1,ν_1,τ_1) is case 1, 2 or 3. In these three cases, we claim that

$$i_{G_2,G_1}\circ i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)\cong i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2).$$

For case 1, recall from [B-J1] that $(P_2,\nu_2,\tau_2)=(c\cdot P_1,c\cdot\nu_1,c\cdot\tau_1)$, for some $c\in C_2$. By Lemma 4.4 [B-J1] (noting that the "not" in its proof is a typo),

$$i_{G_1,c\cdot M_1}(exp\,c\cdot\nu_1\otimes c\cdot\tau_1)\cong c\cdot i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1).$$

Therefore,

$$\begin{aligned} i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2) &\cong i_{G_2,G_1}\circ i_{G_1,c\cdot M_1}(exp\,c\cdot\nu_1\otimes c\cdot\tau_1)\\ &\cong i_{G_2,G_1}\circ c\circ i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)\\ &\cong i_{G_2,G_1}\circ i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1), \end{aligned}$$

as claimed. For case 2, $(P_2,\nu_2,\tau_2)=(P_1,c\cdot\nu_1,c\cdot\tau_1)$, for some $c\in C_2$. The argument is almost identical to that for case 1; we omit the details. For case 3, $(P_2,\nu_2,\tau_2)=(P_{\Phi,C_2(\Phi,\nu_1)},\nu_1,i_{M_{\Phi,C_2(\Phi,\nu_1)},M_1}(\tau_1))$. In this case,

$$\begin{aligned} i_{G_2,M_{\Phi,C_2(\Phi,\nu_1)}}(exp\,\nu_1\otimes i_{M_{\Phi,C_2(\Phi,\nu_1)},M_1}(\tau_1)) &\cong i_{G_2,M_1}(exp\,\nu_1\otimes\tau_1)\\ &\cong i_{G_2,G_1}\circ i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1), \end{aligned}$$

as needed.

To see multiplicity one for cases 1, 2 and 3, write

$$i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)=\pi_1+\sum_i\theta_1^{(i)}$$

in the Grothendieck group. From the preceding claim,

$$i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2)=i_{G_2,G_1}(\pi_1)+\sum_i i_{G_2,G_1}(\theta_1^{(i)}).$$

Observe that $i_{G_2,G_1}(\pi_1) \cong \pi_2$. Theorem 3.4 for G_1 tells us that $\pi_1 \neq \theta_1^{(i)}$. Further, $c \cdot \pi_1 \not\cong \pi_1$ is not a subquotient of $i_{G_2,M_1}(exp\,\nu_1 \otimes \tau_1)$. Therefore, π_2 appears with multiplicity one in $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$.

We now consider the remaining case, that is, when (P_1, ν_1, τ_1) is case 4. In this case, we have

$$i_{G_2,M_1}(exp\,\nu_1 \otimes \tau_1) \cong i_{G_2,M_{\Phi,C_2(\Phi,\nu_1)}}(exp\,\nu_1 \otimes \bigoplus_{\chi \in \hat{D}} \chi\, \tau_2).$$

Thus, it suffices to show that π_2 appears with multiplicity one in $i_{G_2,M_1}(exp\,\nu_1 \otimes \tau_1)$. Write

$$i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1) = \pi_1 + \sum_i \theta_1^{(i)}.$$

Then,

$$i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2) = i_{G_2,G_1}(\pi_1) + \sum_i i_{G_2,G_1}(\theta_1^{(i)})$$

in the Grothendieck group. Observe that $i_{G_2,G_1}(\pi_1)$ contains one copy of π_2, and π_2 is not a subquotient of $i_{G_2,G_1}(\theta_1^{(i)})$ (by multiplicity one for G_1 and the fact that if θ_1 is irreducible such that π_2 is a subquotient of $i_{G_2,G_1}(\theta_1)$, then $\theta_1 \cong \pi_1$). Therefore, π_2 appears with multiplicity one in $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$, as needed. □

Lemma 3.6. *Assume Theorem 3.4 holds for G_1. Let (P_1, ν_1, τ_1) be a set of Langlands data for G_1, with corresponding Langlands subrepresentation π_1. Let θ_1 be an irreducible subquotient of $i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1)$. Suppose*

$$\pi \leq i_{G_2,G_1}(\pi_1), \quad \theta \leq i_{G_2,G_1}(\theta_1)$$

are irreducible subquotients with Langlands data (P, ν, τ) and $(P_\theta, \nu_\theta, \tau_\theta)$, respectively. Then $C_2 \cdot \nu \leq C_2 \cdot \nu_\theta$. Further, if equality occurs, then θ is a component of $i_{G_2,G_1}(\pi_1)$.

Proof. Let $(P_{\theta_1}, \nu_{\theta_1}, \tau_{\theta_1})$ be the Langlands data for θ_1. From the construction of Langlands data in [B-J1], we have $\nu_\theta = c_\theta \cdot \nu_{\theta_1}$ and $\nu = c_\pi \cdot \nu_1$, for some $c_\theta, c_\pi \in C_2$. Since θ_1 is a subquotient of $i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1)$, Theorem 3.4 for G_1 tells us $C_1 \cdot \nu_1 < C_1 \cdot \nu_{\theta_1}$ or $\nu_1 = \nu_{\theta_1}$, with equality if and only if $\theta_1 \cong \pi_1$. Therefore, $C_2 \cdot \nu \leq C_2 \cdot \nu_\theta$, as claimed. Further, if equality occurs, then $\theta \leq i_{G_2,G_1}(\theta_1) \cong i_{G_2,G_1}(\pi_1)$. □

Lemma 3.7. *Assume Theorem 3.4 holds for G_1. Let (P_2, ν_2, τ_2) be a set of Langlands data for G_2 and π_2 the corresponding Langlands subrepresentation. Suppose θ is an irreducible subquotient of $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$ having Langlands data $(P_\theta, \nu_\theta, \tau_\theta)$. Then $C_2 \cdot \nu_2 < C_2 \cdot \nu_\theta$ or $\nu_2 = \nu_\theta$. Further, if equality occurs, then $\theta \cong \pi_2$.*

Proof. Recall from the proof of the Langlands classification in the non-connected case that there are three possibilities: If $P_1 = P_2 \cap G_1$, we may have

1. $P_2 = P_1$.

2. $P_2 \neq P_1$ and $r_{M_1,M_2}(\tau)$ is reducible.
3. $P_2 \neq P_1$ and $r_{M_1,M_2}(\tau)$ is irreducible.

We break the proof into these three cases.

In case 1, $(P_2, \nu_2, \tau_2) = (P_1, \nu_1, \tau_1)$ is also a set of Langlands data for G_1; let π_1 denote the corresponding Langlands subrepresentation. Then, $\pi_2 \cong i_{G_2,G_1}(\pi_1)$. If θ is an irreducible subquotient of $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$, then

$$\theta \leq i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2) \cong i_{G_2,G_1} \circ i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1),$$

so θ is a subquotient of $i_{G_2,G_1}(\theta_1)$ for some irreducible $\theta_1 \leq i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1)$. Lemma 3.6 implies $C_2 \cdot \nu_2 \leq C_2 \cdot \nu_\theta$. Further, if equality occurs, then $\theta \leq i_{G_2,G_1}(\pi_1) \cong \pi_2$, so $\theta \cong \pi_2$ and $\nu_\theta = \nu_2$.

In the second case, let $\nu_1 = \nu_2$ and let τ_1 be an irreducible subquotient of $r_{M_1,M_2}(\tau_2)$. Then (P_1, ν_1, τ_1) is a set of Langlands data for G_1; let π_1 denote the corresponding Langlands subrepresentation. We have $\pi_2 \cong i_{G_2,G_1}(\pi_1)$. If θ is an irreducible subquotient of $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$, then (noting $i_{M_2,M_1}(\tau_1) \cong \tau_2$)

$$\begin{aligned}\theta &\leq i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2) \cong i_{G_2,M_2}(exp\,\nu_2 \otimes i_{M_2,M_1}(\tau_1)) \\ &\cong i_{G_2,M_1}(exp\,\nu_1 \otimes \tau_1) \cong i_{G_2,G_1} \circ i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1),\end{aligned}$$

so θ is a subquotient of $i_{G_2,G_1}(\theta_1)$ for some irreducible $\theta_1 \leq i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1)$. The result then follows from Lemma 3.6 in the same way as in case 1.

In case 3, let $\nu_1 = \nu_2$ and $\tau_1 = r_{M_1,M_2}(\tau_2)$. Then (P_1, ν_1, τ_1) is a set of Langlands data for G_1; let π_1 denote the corresponding Langlands subrepresentation. Then,

$$i_{G_2,G_1}(\pi_1) = \bigoplus_{\chi \in \hat{D}} \chi\, \pi_2.$$

Observe that

$$i_{G_2,G_1} \circ i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1) \cong i_{G_2,M_2}(exp\,\nu_2 \otimes \bigoplus_{\chi \in \hat{D}} \chi\, \tau_2).$$

Write $i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1) = \pi_1 + \sum_i \theta_1^{(i)}$ in the Grothendieck group. Then

$$\begin{aligned}i_{G_2,G_1} \circ i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1) &= i_{G_2,G_1}(\pi_1) + \textstyle\sum_i i_{G_2,G_1}(\theta_1^{(i)}) \\ &= \textstyle\sum_{\chi \in \hat{D}} \chi\, \pi_2 + \sum_i i_{G_2,G_1}(\theta_1^{(i)}).\end{aligned}$$

If $\chi\pi_2 \leq i_{G_2,G_1}(\theta_1^{(i)})$, then $\pi_1 = \theta_1^{(i)}$, which contradicts multiplicity one for G_1. Thus, for each $\chi \in \hat{D}$, we have $\chi\, \pi_2$ occurs with multiplicity one in $i_{G_2,G_1} \circ i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1)$, hence with multiplicity one in $\bigoplus_{\chi \in \hat{D}} i_{G_2,M_2}(exp\,\nu_2 \otimes \chi\, \tau_2)$. Now,

$$i_{G_2,M_2}(exp\,\nu_2 \otimes \chi\, \tau_2) \cong \chi \circ i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$$

contains $\chi\, \pi_2$ with multiplicity one. In particular, this means $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$ contains π_2 with multiplicity one and no $\chi\, \pi_2$ with $\chi \neq 1$.

Let θ be an irreducible subquotient of $i_{G_2,M}(exp\,\nu \otimes \tau)$. By Lemma 3.6, $C_2 \cdot \nu_2 \leq C_2 \cdot \nu_\theta$. Further, if $C_2 \cdot \nu_2 = C_2 \cdot \nu_\theta$ (in the ordering), then θ is a subquotient of $i_{G_2,G_1}(\pi_1)$, so $\theta \cong \chi\,\pi_2$, for some χ. Since $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$ contains no $\chi\,\pi_2$ with $\chi \neq 1$, it follows that $\theta \cong \pi_2$ and $\nu_\theta = \nu_2$. □

Theorem 3.4 now follows from Lemmas 3.5, 3.7, and induction. □

Corollary 3.8. *In the Grothendieck group, any irreducible representation may be written as a linear combination of standard induced representations, i.e., of representations having the form* $i_{G,M}(exp\,\nu \otimes \tau)$ *with* (P, ν, τ) *Langlands data.*

Remark 3.1. *As noted in Remark 4.2 [B-J1], the Langlands classification may also be formulated in the quotient setting. In this setting, Theorem 3.4 and Corollary 3.8 hold, with only the change* $C \cdot \nu > C \cdot \theta$ *in Theorem 3.4 2. required.*

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