

THE AUBERT INVOLUTION AND R-GROUPS

L'INVOLUTION D'AUBERT ET LES R-GROUPES

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ABSTRACT. We establish the connection between the standard intertwining operators for a square integrable representation and its Aubert involution. In the cases of special orthogonal groups and symplectic groups, we define the R-group for a nontempered unitary representation, under the assumption that the Aubert involution of the representation is square-integrable.

RÉSUMÉ. Nous établissons le lien entre les opérateurs d'entrelacement standard associés à une série discrète et à son involution d'Aubert. Dans le cas des groupes spéciaux orthogonaux et des groupes symplectiques, nous définissons le R-groupe pour toute représentation unitaire non-tempérée dont l'involution d'Aubert est une série discrète, sous l'hypothèse que l'involution d'Aubert de cette représentation est une série discrète.

1. INTRODUCTION

This work is on defining the R-group for a nontempered unitary representation of a connected reductive p -adic group, under the assumption that the Aubert involution of the representation is square-integrable.

The R -group determines the reducibility of the induced representation and plays an important role in the trace formula. Classically, the R-group is defined in terms of the Plancherel measure and hence requires temperedness. An alternate description of the R-group is in terms of the L-group and the Langlands correspondence ([A1]). Arthur conjectured that in this context, one should be able to define an R-group, with right basic properties, for certain cases of nontempered representations.

L^AT_EX

Jantzen in [J1] used Iwahori-Matsumoto involution to define the R -group for some unramified principal series representations. Our main tool is the Aubert involution ([Au]). This involution maps an irreducible representation into an irreducible one. Also, it commutes with parabolic induction at the level of Grothendieck groups. The representation and its Aubert involution share supercuspidal support and this implies the connection between standard intertwining operators (Lemma 7.1).

Let G be a split connected reductive p -adic group, $P = MU$ a standard parabolic subgroup. Let σ be a square integrable representation of M . Denote by $\hat{\sigma}$ the Aubert involution of σ . Suppose that $\hat{\sigma}$ unitary. Let R denote the R -group corresponding to $i_{G,M}(\sigma)$. Attached to each element $r \in R$ is the normalized standard intertwining operator

$$A(\sigma, r) \in \text{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma)) = C(\sigma).$$

The set $\{A(\sigma, r) \mid r \in R\}$ is a basis for $C(\sigma)$.

We prove that $i_{G,M}(\sigma)$ and $i_{G,M}(\hat{\sigma})$ have the same intertwining algebras, i.e.,

$$\begin{aligned} \text{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma)) &\cong \text{Hom}_G(i_{G,M}(\hat{\sigma}), i_{G,M}(\hat{\sigma})), \text{ i.e.,} \\ C(\sigma) &\cong C(\hat{\sigma}) \end{aligned}$$

(Corollary 3.4). Knowing that $C(\hat{\sigma})$ is isomorphic to $C(\sigma)$, we would like to have a basis for $C(\hat{\sigma})$ consisting of standard intertwining operators. We establish the connection between the normalized intertwining operators

$$A(\sigma, r) \longleftrightarrow A(\hat{\sigma}, r),$$

which is a consequence of the relation between the standard intertwining operators

$$\mathbf{A}(\nu, \sigma, r) \longleftrightarrow \mathbf{A}(\nu, \hat{\sigma}, r).$$

In the rank-one case, we prove that $\mathbf{A}(\nu, \sigma, r)$ is holomorphic at $\nu = 0$ if and only if $\mathbf{A}(\nu, \hat{\sigma}, r)$ is holomorphic at $\nu = 0$ (Lemma 7.1). Consequently, $A(\sigma, r)$ is a scalar if and only if $A(\hat{\sigma}, r)$ is scalar (Lemma 7.1). Moreover, $A(\sigma, r)$ (respectively, $A(\hat{\sigma}, r)$) is non-scalar if and only if $i_{G,M}(\sigma)$ (respectively, $i_{G,M}(\hat{\sigma})$) is reducible. In that case, $i_{G,M}(\sigma)$ (respectively, $i_{G,M}(\hat{\sigma})$) has length two and $A(\sigma, r)$ (respectively, $A(\hat{\sigma}, r)$) acts on one irreducible component as multiplication by 1 and on another irreducible component as multiplication by -1 (Lemma 5.2). Generally, we have factorizations of $A(\sigma, r)$ and $A(\hat{\sigma}, r)$ such that each factor is induced by an intertwining operator for a rank-one subgroup (Corollary 6.3).

In the case when G is the special orthogonal group or symplectic group, explicit description of R -groups (for square-integrable representations) is given by Goldberg in [G]. Using his result and the relation between standard intertwining operators for σ and $\hat{\sigma}$, we were able to prove that σ and $\hat{\sigma}$ have the same R -group, i.e., that the set

$$\{A(\hat{\sigma}, r) \mid r \in R\}$$

is a basis for $C(\hat{\sigma})$ (Theorem 8.1).

We now give a short summary of the paper. In the second section, we give notation and preliminaries. In the third section, we prove that $C(\sigma) \cong C(\hat{\sigma})$. The fourth section is about the Aubert involution of subrepresentations of $i_{G,M}(\sigma)$. In the fifth section, we use the induction by intertwining operators to describe standard and normalized operators. The sixth section gives a factorization of intertwining operators. The seventh section describes the relation between standard intertwining operators for σ and $\hat{\sigma}$. In the eighth section, we consider symplectic and special orthogonal group and prove that σ and $\hat{\sigma}$ have the same R -group.

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2. PRELIMINARIES

In this section, we shall introduce basic notation and recall some results that will be needed in the rest of the paper.

Let G be a split connected reductive p -adic group. We fix a maximal split torus A_θ of G and a minimal parabolic subgroup P_θ which has A_θ as its split component. We denote by $W = W(G/A_\theta)$ the Weyl group of G with respect to A_θ .

Denote by Σ the set of roots of G with respect to A_θ . The choice of P_θ determines a basis Δ of Σ (which consists of simple roots). It also determines the set of positive roots Σ^+ and the set of negative roots Σ^- .

Let Θ be a subset of Δ . We define Σ_Θ to be the subset of roots in the linear span of Θ . Then

$$\Sigma_\Theta = \Sigma_\Theta^+ \cup \Sigma_\Theta^-,$$

where $\Sigma_\Theta^+ = \Sigma^+ \cap \Sigma_\Theta$, $\Sigma_\Theta^- = \Sigma^- \cap \Sigma_\Theta$.

Let $P_\Theta = M_\Theta U_\Theta$ be the standard parabolic subgroup corresponding to $\Theta \subset \Delta$. For $\alpha \in \Sigma$, let U^α be the corresponding root group ([Bo],

Theorem 13.18). Then

$$\begin{aligned} A_\Theta &= \bigcap_{\alpha \in \Theta} \text{Ker} \alpha, & M_\Theta &= Z_G(A_\Theta), \\ U_\Theta &= \prod_{\alpha \in \Sigma^+ - \Sigma_\Theta^+} U^\alpha, & U_\Theta^- &= \prod_{\alpha \in \Sigma^- - \Sigma_\Theta^-} U^\alpha. \end{aligned}$$

We denote by $W_\Theta = W(M_\Theta/A_\Theta)$ the Weyl group of M_Θ with respect to A_Θ .

Let $M = M_\Theta$ be the standard Levi subgroup of G corresponding to Θ and let $A = A_\Theta$. Denote by $X(M)_F$ and $X(A)_F$ respectively the group of all F -rational characters of M and A . Let

$$\mathfrak{a} = \text{Hom}(X(M)_F, \mathbb{R}) = \text{Hom}(X(A)_F, \mathbb{R})$$

be the real Lie algebra of A and

$$\mathfrak{a}^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} = X(A)_F \otimes_{\mathbb{Z}} \mathbb{R}$$

its dual. Set

$$\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes \mathbb{C}.$$

There is a homomorphism (cf. [H-C]) $H_M : M \rightarrow \mathfrak{a}$ such that

$$q^{\langle \chi, H_M(m) \rangle} = |\chi(m)|$$

for all $m \in M$, $\chi \in X(M)_F$. Given $\nu \in \mathfrak{a}^*$, let us write

$$\exp \nu = q^{\langle \nu, H_M(\cdot) \rangle}$$

for the corresponding character.

Let $\Sigma(\Theta)$ be the set of all the roots of (P_Θ, A_Θ) . For $\alpha \in \Sigma$, we define α_Θ to be the restriction of α to A_Θ . Then $\Sigma(\Theta) = \{\alpha_\Theta \mid \alpha \in \Sigma - \Sigma_\Theta\}$. Given $\alpha \in \Sigma - \Sigma_\Theta$, let

$$[\alpha] = \{\gamma \in \Sigma - \Sigma_\Theta \mid \gamma_\Theta = \alpha_\Theta\}.$$

For $\Theta, \Theta' \subseteq \Delta$, we define

$$W(\Theta, \Theta') = \{w \in W \mid w\Theta' = \Theta\}.$$

We say that Θ and Θ' are associated ([C]) if the set $W(\Theta, \Theta')$ is not empty. If $\Theta = \Theta'$, then we set $W(\Theta) = W(\Theta, \Theta)$ and observe that this is a subgroup of W .

We denote by $\text{Alg}G$ the category of all smooth ([C]) representations of G . We will describe the functors

$$\begin{aligned} i_{G,M} &: \text{Alg}M \rightarrow \text{Alg}G, \\ r_{M,G} &: \text{Alg}G \rightarrow \text{Alg}M, \end{aligned}$$

the functor of parabolic induction and Jacquet functor ([BZ], [C]).

Let $(\sigma, V) \in \text{Alg}M$. Denote by $i_{G,M}(V)$ the set of all smooth functions $f : G \rightarrow V$ satisfying $f(umg) = \delta_P^{1/2}(m)\sigma(m)f(g)$, for all $u \in U, m \in M, g \in G$. (Here δ_P denotes the module of P .) Then $i_{G,M}(\sigma)$ is the representation of G on $i_{G,M}(V)$ defined by

$$(i_{G,M}(\sigma)(g)f)(x) = f(xg), \quad f \in i_{G,M}(V), \quad x, g \in G$$

(G acts on $i_{G,M}(V)$ by right translations). By abuse of notation, we will denote $i_{G,M}(V)$ also by $i_{G,M}(\sigma)$.

If $\varphi \in \text{Hom}_M(\sigma_1, \sigma_2)$, then the induced intertwining operator $i_{G,M}(\varphi) \in \text{Hom}_G(i_{G,M}(\sigma_1), i_{G,M}(\sigma_2))$ is defined by

$$i_{G,M}(\varphi)(f) = \varphi \circ f.$$

Let $(\pi, V) \in \text{Alg}G$. Let

$$V(U) = \text{span}_{\mathbb{C}}\{\pi(u)v - v \mid u \in U, v \in V\}.$$

The representation $r_{M,G}(\pi) \in \text{Alg}M$ is defined on the space $r_{M,G}(V) = V/V(U)$ by

$$r_{M,G}(\pi)(m)(v + V(U)) = \delta_P^{-1/2}(m)\pi(m)v + V(U).$$

If φ is an intertwining operator on π , then $\varphi(V(U)) = V(U)$. The intertwining operator $r_{M,G}(\varphi) : r_{M,G}(V) \rightarrow r_{M,G}(V)$ is defined by

$$r_{M,G}(\varphi)(v + V(U)) = \varphi(v) + V(U).$$

Let $R(G)$ be the Grothendieck group of the category of all smooth finite length representations of G . For a smooth finite length representation π of G , we define $s.s.(\pi) \in R(G)$ to be the sum of the irreducible components of π , each component taken with the multiplicity corresponding to its multiplicity in π . Let $\pi_1, \pi_2 \in R(G)$. We write $\pi_1 \leq \pi_2$ if, for each irreducible component ρ of π_1 , the multiplicity of ρ in π_1 is less than or equal to the multiplicity of ρ in π_2 . For smooth finite length representations π_1 and π_2 , we write $\pi_1 \leq \pi_2$ if $s.s.(\pi_1) \leq s.s.(\pi_2)$ in the Grothendieck group.

$i_{G,M}$ and $r_{M,G}$ induce functors

$$\begin{aligned} i_{G,M} : R(M) &\rightarrow R(G), \\ r_{M,G} : R(G) &\rightarrow R(M). \end{aligned}$$

Fix two associated subsets Θ and Θ' of Δ . Let (σ, V) be an irreducible admissible representation of M . For $w \in W(\Theta, \Theta')$, set $U_w = U_{\emptyset} \cap wU^-w^{-1}$. Let $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ and $f \in i_{G,M}(\exp \nu \otimes \sigma)$. We formally define

$$\mathbf{A}(\nu, \sigma, w)f(g) = \int_{U_w} f(w^{-1}ug)du,$$

where $g \in G$. If $\mathbf{A}(\nu, \sigma, w)$ converges, it defines an intertwining operator between $i_{G,M}(\exp \nu \otimes \sigma)$ and $i_{G,M}(\exp w\nu \otimes w\sigma)$. We call it a standard intertwining operator. We also denote $\mathbf{A}(\nu, \sigma, w)$ by $\mathbf{A}_G(\nu, \sigma, w)$.

We refer to [A2], [GSh], [K] and [Sh1] for properties of standard intertwining operators. We need the following property:

if $l(w_2w_1) = l(w_2) + l(w_1)$, then

$$\mathbf{A}(\nu, \sigma, w_2w_1) = \mathbf{A}(w_1\nu, w_1\sigma, w_2)\mathbf{A}(\nu, \sigma, w_1)$$

([GSh], Proposition 2.13, [A2], page 26). We define a normalized intertwining operator

$$A'(\nu, \sigma, w) = n(\nu, \sigma, w)\mathbf{A}(\nu, \sigma, w),$$

where $n(\nu, \sigma, w)$ is a normalizing factor. The existence of normalizing factors for square integrable representations was proved by Harish-Chandra ([S2]). Shahidi ([Sh2]) described normalizing factors in terms of L -functions and root numbers. We refer to [A2], Theorem 2.1, for the proof of the existence of normalizing factors for any irreducible admissible representation of G .

Set

$$\begin{aligned} \mathbf{A}(\sigma, w) &= \mathbf{A}(0, \sigma, w), \\ A'(\sigma, w) &= A'(0, \sigma, w). \end{aligned}$$

Let $W(\sigma) = \{w \in W \mid w\sigma \cong \sigma\}$. Here $w\sigma$ is defined in a usual way: $w\sigma(m) = \sigma(w^{-1}mw)$, $m \in M$. For $w \in W(\sigma)$, let $T_w : V \rightarrow V$ be an isomorphism between $w\sigma$ and σ ([GSh]). Define

$$A(\sigma, w) = T_w \circ A'(\sigma, w).$$

This is an isomorphism between $i_{G,M}(\sigma)$ and $i_{G,M}(w\sigma)$. We have

$$A(\sigma, w_2w_1) = \eta(w_2, w_1)A(w_1\sigma, w_2)A(\sigma, w_1),$$

where $\eta(w_2, w_1)$ is given by $T_{w_2w_1} = \eta(w_2, w_1)T_{w_2}T_{w_1}$. In this paper, we shall assume that $\eta(w_2, w_1) = 1$, $\forall w_1, w_2 \in W(\sigma)$. This is known for: $GL(n)$, $SL(n)$, $Sp(2n)$, $SO(n)$, $U(n)$, $GO(n)$, $GSp(2n)$, $GU(n)$, and for all principal series of several other groups.

Theorem 2.1. (*Harish-Chandra [S2]*) *Let σ be an irreducible square integrable representation of M . The set of normalized intertwining operators $\{A(\sigma, w) \mid w \in W(\sigma)\}$ spans the commuting algebra*

$$C(\sigma) = \text{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma)).$$

The R -group is defined for square-integrable representations, using the Plancherel measure. The definition can be found in [A3], [G] or [GSh]. The R -group is a subgroup of $W(\sigma)$ and its basic properties are given by the following theorem:

Theorem 2.2. (*Knapp-Stein, Silberger [KnSt],[S2]*) Let σ be an irreducible square integrable representation of M and let R be the R -group for σ .

1. $W(\sigma) = R \times W'$, where $W' = \{w \mid A(\sigma, w) \text{ is a scalar}\}$.
2. $\{A(\sigma, r) \mid r \in R\}$ is a basis for $C(\sigma)$.

3. THE AUBERT INVOLUTION

We shall prove that, in the Grothendieck group, an irreducible admissible representation and its Aubert involution have isomorphic intertwining algebras (Corollary 3.4).

Theorem 3.1. (*Aubert [Au]*) Define the operator D_G on the Grothendieck group $R(G)$ by

$$D_G = \sum_{\Phi \subset \Delta} (-1)^{|\Phi|} i_{G, M_\Phi} \circ r_{M_\Phi, G}.$$

D_G has the following properties:

1. $D_G \circ \tilde{} = \tilde{} \circ D_G$ (here $\tilde{}$ denotes contragredient).
2. $D_G \circ i_{G, M} = i_{G, M} \circ D_M$.
3. For the standard Levi subgroup $M = M_\Theta$,

$$r_{M, G} \circ D_G = w \circ D_{w^{-1}(M)} \circ r_{w^{-1}(M), G},$$

where w is the longest element of the set $\{w \in W \mid w^{-1}(\Theta) > 0\}$.

4. D_G is an involution, i.e., $D_G^2 = id$.
5. If σ is irreducible supercuspidal, then $D_G(\sigma) = (-1)^{|\Delta|} \sigma$.
6. D_G takes irreducible representations to irreducible representations.

If σ is an irreducible unitary representation of G , we will denote by $\hat{\sigma}$ the representation $\pm D_G(\sigma)$, taking the sign $+$ or $-$ so that $\hat{\sigma}$ is a positive element of $R(G)$. We will call $\hat{\sigma}$ the Aubert involution of σ .

Lemma 3.2. Suppose that $(\pi_1, V_1), \dots, (\pi_n, V_n), (\pi, V)$ are representations of G . Then

1. $Hom_G(\pi_1 \oplus \dots \oplus \pi_n, \pi) \cong Hom_G(\pi_1, \pi) \oplus \dots \oplus Hom_G(\pi_n, \pi)$.
2. $Hom_G(\pi, \pi_1 \oplus \dots \oplus \pi_n) \cong Hom_G(\pi, \pi_1) \oplus \dots \oplus Hom_G(\pi, \pi_n)$.

Proof. The isomorphisms

$$\begin{aligned} Hom_{\mathbb{C}}(V_1 \oplus \dots \oplus V_n, V) &\cong Hom_{\mathbb{C}}(V_1, V) \oplus \dots \oplus Hom_{\mathbb{C}}(V_n, V), \\ Hom_{\mathbb{C}}(V, V_1 \oplus \dots \oplus V_n) &\cong Hom_{\mathbb{C}}(V, V_1) \oplus \dots \oplus Hom_{\mathbb{C}}(V, V_n) \end{aligned}$$

defined in [L], map intertwining operators to intertwining operators and give the lemma. \square

Lemma 3.3. *Suppose that (π, V) and (π', V') are semisimple finite length representations of G . If (π, V) and (π', V') have the same number of irreducible components with the same multiplicities, then $\text{Hom}_G(\pi, \pi)$ and $\text{Hom}_G(\pi', \pi')$ are isomorphic algebras.*

Proof. According to Lemma 3.2,

$$\dim_{\mathbb{C}} \text{Hom}_G(\pi, \pi) = \dim_{\mathbb{C}} \text{Hom}_G(\pi', \pi').$$

This proves the vector spaces isomorphism. To prove isomorphism of algebras, we will show that $\text{Hom}_G(\pi, \pi)$ and $\text{Hom}_G(\pi', \pi')$ have the same multiplication tables.

First, suppose that π is the direct sum of n equivalent representations. Then

$$\begin{aligned} \pi &\cong \sigma \oplus \cdots \oplus \sigma, \\ V &\cong V'' = V_1 \oplus \cdots \oplus V_n, \end{aligned}$$

where V_1, \dots, V_n are n copies of the same space. For $i, j = 1, \dots, n$, define $\varphi_{ij} : V_i \rightarrow V_j$ by

$$\varphi_{ij}(v) = v.$$

Define $\Phi_{ij} : V'' \rightarrow V''$ by

$$\Phi_{ij} = 0 \oplus \cdots \oplus 0 \oplus \varphi_{ij} \oplus 0 \oplus \cdots \oplus 0,$$

where φ_{ij} is on the i -th place. Then $\{\Phi_{ij} \mid i, j = 1, \dots, n\}$ is a set of n^2 linearly independent intertwining operators, so it is a basis for $\text{Hom}_G(\sigma \oplus \cdots \oplus \sigma, \sigma \oplus \cdots \oplus \sigma)$. The multiplication is given by

$$\Phi_{ij} \cdot \Phi_{pq} = \begin{cases} \Phi_{pq}, & \text{if } j = p, \\ 0, & \text{if } j \neq p. \end{cases}$$

Generally, let $\{\sigma_1, \dots, \sigma_k\}$ be the set of equivalence classes of irreducible subquotients of π . Then there exists a decomposition

$$\begin{aligned} \pi &= \pi_1 \oplus \cdots \oplus \pi_k, \\ V &= V_1 \oplus \cdots \oplus V_k, \end{aligned}$$

such that, for every $m = 1, \dots, k$, π_m is the direct sum of representations equivalent to σ_m . Note that for $m \neq l$, $\text{Hom}_G(\pi_m, \pi_l) = \{0\}$.

Let $\{\varphi_{mi}\}_{i \in I_m}$ be a basis for $\text{Hom}_G(\pi_m, \pi_m)$. Define

$$\Phi_{mi} = 0 \oplus \cdots \oplus 0 \oplus \varphi_{mi} \oplus 0 \oplus \cdots \oplus 0,$$

where φ_{mi} is on the m -th place. Then $\{\Phi_{mi} \mid m = 1, \dots, k, i \in I_m\}$ is a basis for $\text{Hom}_G(\pi, \pi)$. If $l \neq m$, then

$$\Phi_{mi} \cdot \Phi_{lj} = 0.$$

□

Corollary 3.4. *Let σ be an irreducible admissible representation of M , $\hat{\sigma}$ be the Aubert involution of σ . Then*

$$\text{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma)) \cong \text{Hom}_G(i_{G,M}(\hat{\sigma}), i_{G,M}(\hat{\sigma}))$$

in the Grothendieck group $R(G)$.

Proof. It follows from Theorem 3.1, 2., that $i_{G,M}(\hat{\sigma}) = \widehat{i_{G,M}(\sigma)}$. Therefore, $i_{G,M}(\sigma)$ and $i_{G,M}(\hat{\sigma})$ have the same number of irreducible components. The multiplicities are the same because $\hat{\cdot}$ is an involution (Theorem 3.1, 4.). \square

4. THE AUBERT INVOLUTION OF SUBREPRESENTATIONS OF $i_{G,M}(\sigma)$

Let $P = MU$ be the standard parabolic subgroup corresponding to $\Theta \subset \Delta$ and σ an irreducible supercuspidal representation of M . If p is an irreducible subrepresentation of $i_{G,M}(\sigma)$, we will prove that $i_{G,M}(\sigma)$ has a quotient equivalent to \hat{p} (Corollary 4.2).

Denote by P^- the opposite parabolic subgroup of P , i.e., the unique parabolic subgroup intersecting P in M . Let $\bar{P} = \bar{M}\bar{U}$ be the unique standard parabolic subgroup conjugate to P^- ([C]); we can have either $\bar{P} = P$ or $\bar{P} \neq P$.

Let w_l denote the longest element in the Weyl group and $w_{l,\Theta}$ the longest element in $W_\Theta = W(M/A_\Theta)$. Then $w = w_l w_{l,\Theta}$ is the longest element in the set $\{w \in W \mid w(\Theta) > 0\}$ ([C]).

Set $\bar{\Theta} = w(\Theta)$. Then \bar{P} corresponds to $\bar{\Theta}$. For the unipotent radical U^- of P^- , we have

$$w(U^-) = \bar{U}.$$

According to [C], Proposition 1.3.3, $w(M_\Theta) = M_{w(\Theta)}$, so

$$w(M) = w(M_\Theta) = M_{w(\Theta)} = M_{\bar{\Theta}} = \bar{M}.$$

Lemma 4.1. *Let σ be an irreducible supercuspidal representation of M .*

1. *An irreducible representation p is equivalent to a subrepresentation of $i_{G,M}(\sigma)$ if and only if $\sigma \leq r_{M,G}(p)$.*
2. *An irreducible representation q is equivalent to a quotient of $i_{G,M}(\sigma)$ if and only if $w\sigma \leq r_{\bar{M},G}(q)$, where $w = w_l w_{l,\Theta}$.*

In the proof of Lemma 4.1, we shall use Jacquet modules with respect to non-standard parabolic subgroups, in the notation of [BeZ]: if $P = MU$ is a parabolic subgroup of G , and π is a representation of G , we denote by $r_{U,1}(\pi)$ the Jacquet module of π with respect to $P = MU$. If P is a standard parabolic subgroup, then $r_{U,1}(\pi) = r_{M,G}(\pi)$.

Proof. 1. Follows from [B1], Corollary 4.3.

2. Let q be an irreducible quotient of $i_{G,M}(\sigma)$. Denote by \tilde{q} the contragredient representation of q . By [C], Proposition 2.1.11, the functor $\pi \mapsto \tilde{\pi}$ is contravariant and exact, so \tilde{q} is a subrepresentation of $\widetilde{i_{G,M}(\sigma)} = i_{G,M}(\tilde{\sigma})$ ([C], Proposition 3.1.2). By 1., $\tilde{\sigma} \leq r_{M,G}(\tilde{q})$ and taking the contragredient, we obtain $\tilde{\tilde{\sigma}} \leq \widetilde{r_{M,G}(\tilde{q})}$. It follows from [C], Proposition 2.1.10 that $\tilde{\tilde{\sigma}} = \sigma$, so $\sigma \leq \widetilde{r_{M,G}(\tilde{q})}$. According to [C], Corollary 4.2.5 and [B1], Corollary 3.4, we have the following

$$\sigma \leq \widetilde{r_{M,G}(\tilde{q})} = r_{U^-,1}(q) = r_{w^{-1}(\bar{U}),1}(q) = w^{-1}(r_{\bar{M},G}(q)).$$

Hence, $w\sigma \leq r_{\bar{M},G}(q)$.

Now, suppose that q is an irreducible representation such that $w\sigma \leq r_{\bar{M},G}(q)$. Then, applying the same arguments as above, we obtain $\tilde{\sigma} \leq r_{M,G}(\tilde{q})$. It follows by 1. that \tilde{q} is equivalent to a subrepresentation of $i_{G,M}(\tilde{\sigma})$. Hence, q is equivalent to a quotient of $i_{G,M}(\sigma)$. \square

Corollary 4.2. *Let σ be an irreducible supercuspidal representation of M . If p is an irreducible subrepresentation of $i_{G,M}(\sigma)$, then $i_{G,M}(\sigma)$ has a quotient equivalent to \hat{p} .*

Proof. Let p be an irreducible subrepresentation of $i_{G,M}(\sigma)$. Then $\sigma \leq r_{M,G}(p)$. Note that w is the longest element in the set $\{w' \in W \mid (w')^{-1}(\bar{\Theta}) > 0\}$.

The representation σ is supercuspidal and p is an irreducible subrepresentation of the induced representation $i_{G,M}(\sigma)$. Therefore, the Jacquet module $r_{w^{-1}(\bar{M}),G}(p)$ is supercuspidal. By Theorem 3.1, 5., the duality operator $D_{w^{-1}(\bar{M})}$ acts as multiplication by 1 or -1. Also, it acts on all irreducible components of $r_{w^{-1}(\bar{M}),G}(p)$ in the same way. The operator $\hat{\cdot}$ is defined as $\pm D$, to assure that we obtain a positive element in the Grothendieck group. Therefore,

$$\widehat{r_{w^{-1}(\bar{M}),G}(p)} = r_{w^{-1}(\bar{M}),G}(p).$$

By Theorem 3.1 and the equality above, we have

$$r_{\bar{M},G}(\hat{p}) = w \circ r_{w^{-1}(\bar{M}),G}(p) = w \circ r_{M,G}(p).$$

It follows that $w\sigma \leq r_{w(M),G}(\hat{p})$. Now, Lemma 4.1 tells us that \hat{p} is equivalent to a quotient of $i_{G,M}(\sigma)$. \square

5. STANDARD INTERTWINING OPERATORS

In this section, we describe standard and normalized intertwining operators using the parabolic induction by intertwining operators.

Lemma 5.1. *Let π be an irreducible admissible representation of M . Let β be a simple root in $\Delta \setminus \Theta$ and $\Omega = \Theta \cup \{\beta\}$. Let $w = w_{l,\Omega}w_{l,\Theta}$. Then*

$$\mathbf{A}_G(\nu, \pi, w) = i_{G, M_\Omega}(\mathbf{A}_{M_\Omega}(\nu, \pi, w)).$$

Notice that $w(\Theta) > 0$, so Θ and $w(\Theta)$ are associated.

Proof. Set $N = M_\Omega$. Let U'_\emptyset denote the the unipotent radical of the minimal parabolic subgroup in N . Then

$$U'_\emptyset = \prod_{\alpha \in \Sigma_\Omega^+} U^\alpha.$$

The standard intertwining operator $\mathbf{A}_N(\nu, \pi, w)$ is defined as an integral over the set

$$U'_w = U'_\emptyset \cap w(U^- \cap N)w^{-1}$$

and $\mathbf{A}_G(\nu, \pi, w)$ is defined by integration over $U_w = U'_\emptyset \cap wU^-w^{-1}$. First, we will prove that

$$U'_w = U_w = w(U^- \cap N)w^{-1}.$$

We have

$$\begin{aligned} U &= \prod_{\alpha \in \Sigma^+ \setminus \Sigma_\Theta^+} U^\alpha, & U^- &= \prod_{\alpha \in \Sigma^- \setminus \Sigma_\Theta^-} U^\alpha, \\ U \cap N &= \prod_{\alpha \in \Sigma_\Omega^+ \setminus \Sigma_\Theta^+} U^\alpha, & U^- \cap N &= \prod_{\alpha \in \Sigma_\Omega^- \setminus \Sigma_\Theta^-} U^\alpha. \end{aligned}$$

If $\alpha \in \Sigma_\Theta^+$, then $w(\alpha) > 0$. It follows that

$$U_w = \prod_{\substack{\alpha \in \Sigma_\Omega^+ \\ w(\alpha) < 0}} U^{w(-\alpha)}, \quad U'_w = \prod_{\substack{\alpha \in \Sigma_\Omega^+ \\ w(\alpha) < 0}} U^{w(-\alpha)}.$$

According to [Bou], Corollary 4, page 20, the length of w in W is equal to the length of w in $W_\Omega = W(M_\Omega/A_0)$. Therefore,

$$\{\alpha \in \Sigma^+ \mid w(\alpha) < 0\} = \{\alpha \in \Sigma_\Omega^+ \mid w(\alpha) < 0\}$$

and $U'_w = U_w$. Now, $w(\alpha) < 0$ if and only if $\alpha \in \Sigma_\Omega^+ \setminus \Sigma_\Theta^+$. It follows

$$U'_w = \prod_{\alpha \in \Sigma_\Omega^+ \setminus \Sigma_\Theta^+} U^{w(-\alpha)} = w\left(\prod_{\alpha \in \Sigma_\Omega^+ \setminus \Sigma_\Theta^+} U^{-\alpha}\right)w^{-1} = w(U^- \cap N)w^{-1}.$$

Hence,

$$\mathbf{A}_N(\nu, \pi, w)f(n) = \int_{U_w} f(w^{-1}un)du,$$

for $f \in i_{N,M}(V)$, $n \in N$, and

$$\mathbf{A}_G(\nu, \pi, w)f(g) = \int_{U_w} f(w^{-1}ug)du,$$

for $f \in i_{G,M}(V)$, $g \in G$. Isomorphisms

$$\begin{aligned} i_{G,M}(\exp \nu \otimes \sigma) &\xrightarrow{\varphi} i_{G,N} \circ i_{N,M}(\exp \nu \otimes \sigma) \\ i_{G,M}(\exp \nu \otimes \sigma) &\xleftarrow{\psi} i_{G,N} \circ i_{N,M}(\exp \nu \otimes \sigma) \end{aligned}$$

are defined by

$$\begin{aligned} F &\xrightarrow{\varphi} F_0, \\ F_0 &\xrightarrow{\psi} F, \end{aligned}$$

$$\begin{aligned} F(g) &= F_0(g)(1), \\ F_0(g)(n) &= \delta_\Omega^{-1/2}(n)F_0(g)(n). \end{aligned}$$

Also,

$$i_{G,N}(\mathbf{A}_N(\nu, \pi, w))F_0 = \mathbf{A}_N(\nu, \pi, w) \circ F_0.$$

Now, we have

$$\begin{aligned} i_{G,N}(\mathbf{A}_N(\nu, \pi, w))F(g) &= i_{G,N}(\mathbf{A}_N(\nu, \pi, w))F_0(g)(1) \\ &= \mathbf{A}_N(\nu, \pi, w) \circ F_0(g)(1) \\ &= \int_{U_w} F_0(g)(w^{-1}u)du = \int_{U_w} \delta_\Omega^{-1/2}(w^{-1}u)F(w^{-1}ug)du \\ &= \int_{U_w} F(w^{-1}ug)du = \mathbf{A}_G(\nu, \pi, w)F(g), \end{aligned}$$

since $\delta_\Omega(w^{-1}u) = \delta_\Omega(w^{-1})\delta_\Omega(u) = 1$.

□

Remark 5.1. Let (π, V) be a semisimple representation of the length two. Hence,

$$\begin{aligned} \pi &= \pi_1 \oplus \pi_2, \\ V &= V_1 \oplus V_2. \end{aligned}$$

Suppose that $\pi_1 \not\cong \pi_2$. Then the decomposition $V = V_1 \oplus V_2$ is unique, $\dim_{\mathbb{C}} \text{Hom}_G(\pi, \pi) = 2$. Let $\varphi : V \rightarrow V$ be an intertwining operator. If $\varphi(V_1) \neq 0$, then $\varphi(\pi_1) \cong \pi_1$, so $\varphi(V_1) = V_1$ and there exists $a \in \mathbb{C}$, $a \neq 0$, such that $\varphi(v_1) = av_1$, for every $v_1 \in V_1$. We write $\varphi|_{V_1} = a$.

Now, if $\varphi : V \rightarrow V$ is an isomorphism, we have

$$\begin{aligned} \varphi|_{V_1} &= a, \\ \varphi|_{V_2} &= b, \end{aligned}$$

where a, b are nonzero complex numbers. Suppose that φ is not equal to a scalar and that $\varphi^2 = 1$. Then $a \neq b$, $a^2 = 1$, $b^2 = 1$. It follows

$$a = 1, b = -1 \quad \text{or} \quad a = -1, b = 1.$$

Note that $\{1, \varphi\}$ is a basis for $\text{Hom}_G(\pi, \pi)$.

Lemma 5.2. *Let α be a simple root in $\Delta \setminus \Theta$ and $\Omega = \Theta \cup \{\alpha\}$. Set $w_\alpha = w_{l_\Omega} w_{l_\Theta}$. Let (σ, V) be an irreducible square integrable representation of $M = M_\Theta$. Suppose that $i_{M_\Omega, M}(\sigma)$ is reducible. Write*

$$i_{M_\Omega, M}(V) = V_1 \oplus V_2,$$

where V_1, V_2 are irreducible and $V_1 \not\cong V_2$. Then (up to exchange of indices)

$$\begin{aligned} A_{M_\Omega}(\sigma, w_\alpha)|_{V_1} &= 1, \\ A_{M_\Omega}(\sigma, w_\alpha)|_{V_2} &= -1. \end{aligned}$$

Further (up to exchange of indices),

$$\begin{aligned} A_G(\sigma, w_\alpha)|_{i_{M_\Omega, M}(V_1)} &= 1, \\ A_G(\sigma, w_\alpha)|_{i_{M_\Omega, M}(V_2)} &= -1 \end{aligned}$$

and

$$A_G(\sigma, w_\alpha) = \pm i_{G, M_\Omega}(A_{M_\Omega}(\sigma, w_\alpha)).$$

Proof. Set $N = M_\Omega$. According to Remark 5.1, there exist $a, b \in \mathbb{C}$, $a \neq b$, such that

$$\begin{aligned} T_{w_\alpha} \circ \mathbf{A}_N(\sigma, w_\alpha)|_{V_1} &= a, \\ T_{w_\alpha} \circ \mathbf{A}_N(\sigma, w_\alpha)|_{V_2} &= b. \end{aligned}$$

Let c be a normalizing factor for $\mathbf{A}_N(\sigma, w_\alpha)$. Then (Remark 5.1),

$$ac = 1, bc = -1 \quad \text{or} \quad ac = -1, bc = 1.$$

We conclude that $a = -b$ and that c is unique up to ± 1 . We may assume that

$$\begin{aligned} A_N(\sigma, w_\alpha)|_{V_1} &= 1, \\ A_N(\sigma, w_\alpha)|_{V_2} &= -1. \end{aligned}$$

Let $f \in i_{G, N}(V_1)$, $f \neq 0$ and let d be a normalizing factor for $\mathbf{A}_G(\sigma, w_\alpha)$. We have $w_\alpha^2 = 1$ and $(A_G(\sigma, w_\alpha))^2 = A_G(\sigma, 1) = 1$. Take $g \in G$ such that $0 \neq f(g) = v_1 \in V_1$. Then

$$\begin{aligned} v_1 &= (A_G(\sigma, w_\alpha))^2 f(g) = d^2 (T_{w_\alpha} \circ \mathbf{A}_G(\sigma, w_\alpha))^2 f(g) \\ &= d^2 (T_{w_\alpha} \circ \mathbf{A}_N(\sigma, w_\alpha))^2 \circ f(g) = d^2 (T_{w_\alpha} \circ \mathbf{A}_N(\sigma, w_\alpha))^2(v_1) = d^2 a^2 v_1. \end{aligned}$$

It follows $da = \pm 1$. Hence, d is unique up to ± 1 and we have

$$\begin{aligned} A_G(\sigma, w_\alpha)|_{i_{N, M}(V_{i_1})} &= 1, \\ A_G(\sigma, w_\alpha)|_{i_{N, M}(V_{i_2})} &= -1, \end{aligned}$$

where $\{i_1, i_2\} = \{1, 2\}$. This implies

$$A_G(\sigma, w_\alpha) = \pm i_{G,N}(A_N(\sigma, w_\alpha)).$$

□

6. FACTORIZATION OF STANDARD INTERTWINING OPERATORS

Recall the following result (Lemma 2.1.2 of [Sh1]):

Lemma 6.1. *Suppose $\Theta, \Theta' \subset \Delta$ are associated. Take $w \in W(\Theta, \Theta')$. Then, there exists a family of subsets $\Theta_1, \dots, \Theta_{n+1} \subset \Delta$ such that*

1. $\Theta_1 = \Theta$ and $\Theta_{n+1} = \Theta'$;
2. fix $1 \leq i \leq n$; then there exists a root $\alpha_i \in \Delta \setminus \Theta_i$ such that Θ_{i+1} is the conjugate of Θ_i in $\Omega_i = \Theta_i \cup \alpha_i$;
3. set $w_i = w_{l, \Omega_i} w_{l, \Theta_i}$ in $W(\Theta_i, \Theta_{i+1})$ for $1 \leq i \leq n$; then

$$w = w_n \cdots w_1.$$

Lemma 6.2. *Suppose that $\Theta, \Theta' \subset \Delta$ are associated and fix $w \in W(\Theta, \Theta')$. Write $w = w_n \cdots w_1$ as in Lemma 6.1. Then*

$$l(w) = l(w_n) + \cdots + l(w_1),$$

where l denotes the length in the Weyl group $W = W(G/A_\emptyset)$.

Proof. Let S_1 and S_2 be as in the proof of Lemma 2.1.2. in [Sh1],

$$\begin{aligned} S_1 &= \{[\alpha] \mid \alpha \in \Sigma^+ - \Sigma_{\Theta_1}^+, w(\alpha) < 0\}, \\ S_2 &= \{[\beta] \mid \beta \in \Sigma^+ - \Sigma_{\Theta_2}^+, ww_1^{-1}(\beta) < 0\}. \end{aligned}$$

The condition $w \in W(\Theta_1, \Theta')$ gives $w(\Theta_1) = \Theta' > 0$. Similarly, $ww_1^{-1} \in W(\Theta_2, \Theta')$ gives $ww_1^{-1}(\Theta_2) = \Theta' > 0$ and we can write

$$\begin{aligned} S_1 &= \{[\alpha] \mid \alpha \in \Sigma^+, w(\alpha) < 0\}, \\ S_2 &= \{[\beta] \mid \beta \in \Sigma^+, ww_1^{-1}(\beta) < 0\}. \end{aligned}$$

If $\beta \in S_2$, then $w^{-1}(\beta) \in S_1$ ([Sh1], proof of Lemma 2.1.2). The mapping $[\beta] \mapsto [w^{-1}(\beta)]$ is an injection from S_2 to S_1 and

$$S_1 - w^{-1}(S_2) = \{[\alpha] \mid \alpha \in \Sigma_{\Omega_1}^+ - \Sigma_{\Theta_1}^+\} = \{[\alpha] \mid \alpha \in \Sigma^+, w_1(\alpha) < 0\}.$$

Recall that, for any $w' \in W$, $l(w') = \text{Card}\{\alpha \in \Sigma^+ \mid w'(\alpha) < 0\}$. We conclude that

$$\begin{aligned} l(w) &= \text{Card}\{\alpha \mid [\alpha] \in S_1\}, \\ l(ww_1^{-1}) &= \text{Card}\{\beta \mid [\beta] \in S_2\}, \\ l(w_1) &= \text{Card}\{[\alpha] \mid \alpha \in S_1 - w^{-1}(S_2)\}. \end{aligned}$$

This implies

$$l(w) = l(w_n \cdots w_2) + l(w_1).$$

The lemma follows by induction on n . □

Corollary 6.3. *Suppose that $\Theta, \Theta' \subset \Delta$ are associated and fix $w \in W(\Theta, \Theta')$. Write $w = w_n \cdots w_1$ as in Lemma 6.1. Then*

$$\mathbf{A}(\nu, \pi, w) = \mathbf{A}(\nu_n, \pi_n, w_n) \cdots \mathbf{A}(\nu_1, \pi_1, w_1),$$

where $\nu_1 = \nu$, $\pi_1 = \pi$, $\nu_i = w_{i-1}(\nu_{i-1})$ and $\pi_i = w_{i-1}(\pi_{i-1})$ for $2 \leq i \leq n$.

Remark 6.1. *Theorem 2.1.1 of [Sh1] gives the same factorization as in Corollary 6.3 in the case when $\mathbf{A}(\nu, \pi, w)$ is absolutely convergent.*

Lemma 6.4. *Let M be the standard parabolic subgroup corresponding to $\Theta \subset \Delta$ and σ an irreducible supercuspidal representation of M . Let q be an irreducible subquotient of $i_{G,M}(\sigma)$. Then there exists $w_0 \in W(\Theta)$ such that the standard intertwining operator $\mathbf{A}(\nu, \sigma, w_0)$ is holomorphic at $\nu = 0$ and q is equivalent to a subrepresentation of $i_{G,M}(w_0\sigma)$.*

Proof. Recall ([C], [BeZ]) that

$$s.s.(r_{M,G} \circ i_{G,M}(\sigma)) = \sum_{w \in W(\Theta)} w\sigma$$

and

$$0 \neq r_{M,G}(q) \leq r_{M,G} \circ i_{G,M}(\sigma).$$

Take $w' \in W(\Theta)$ such that $w'\sigma \leq r_{M,G}(q)$. We choose from the set

$$\{w \mid w\sigma \cong w'\sigma\}$$

an element w_0 with the least length. Write $w_0 = w_n \cdots w_1$ as in Lemma 6.1. The condition on the length of w_0 tells us that

$$w_i w_{i-1} \cdots w_1 \sigma \not\cong w_{i-1} \cdots w_1 \sigma, \quad \text{for } 1 \leq i \leq n.$$

Let $\sigma_1 = \sigma$ and $\sigma_i = w_{i-1}(\sigma_{i-1})$ for $2 \leq i \leq n$. Let Θ_i and Ω_i be as in Lemma 6.1. Denote by N_i the standard Levi subgroup corresponding to Ω_i . Note that $\sigma_i \not\cong \sigma_{i+1}$. Let $\nu_1 = \nu$ and $\nu_i = w_{i-1}(\nu_{i-1})$ for $2 \leq i \leq n$. According to [Sh1], proof of the Theorem 3.3.1, $\mathbf{A}_{N_i}(\nu_i, \sigma_i, w_i)$ is holomorphic at $\nu_i = 0$. Lemma 5.1 tells us that $\mathbf{A}(\nu_i, \sigma_i, w_i)$ is holomorphic at $\nu_i = 0$. We can write $\mathbf{A}(\nu, \sigma, w_0)$ as in Corollary 6.3

$$\mathbf{A}(\nu, \sigma, w_0) = \mathbf{A}(\nu_n, \sigma_n, w_n) \cdots \mathbf{A}(\nu_1, \sigma_1, w_1),$$

and conclude that $\mathbf{A}(\nu, \sigma, w_0)$ is holomorphic at $\nu = 0$. It follows from Lemma 4.1 that q is equivalent to a subrepresentation of $i_{G,M}(w_0\sigma)$ because

$$w_0\sigma \cong w'\sigma \leq r_{M,G}(q).$$

□

7. RELATION BETWEEN STANDARD INTERTWINING OPERATORS
FOR σ AND $\hat{\sigma}$

Let M be the standard Levi subgroup of G corresponding to $\Theta \subset \Delta$. Let (σ, V) be an irreducible square integrable representation of M and $\hat{\sigma}$ the Aubert involution of σ . Suppose that $\hat{\sigma}$ is unitary.

There exists $\Theta_0 \subset \Theta$ and an irreducible supercuspidal representation (σ_0, V_0) of $M_0 = M_{\Theta_0}$ such that σ is a subrepresentation of $i_{M, M_0}(\sigma_0)$ (we allow $\Theta_0 = \Theta$). It follows from Corollary 4.2 that $\hat{\sigma}$ is equivalent to a quotient of $i_{M, M_0}(\sigma_0)$. Let $W_{\Theta} = W(M_{\Theta}/A_{\theta})$. According to Lemma 6.4, there exists $w_0 \in W_{\Theta}(\Theta_0) = \{w \in W_{\Theta} \mid w(\Theta_0) = \Theta_0\}$ such that $\mathbf{A}_M(\nu, \sigma_0, w_0)$ is holomorphic at $\nu = 0$ and $\hat{\sigma}$ is equivalent to a subrepresentation of $i_{M, M_0}(w_0\sigma_0)$.

Lemma 7.1. *Let $\Omega = \Theta \cup \{\alpha\}$, for a simple root α , and $N = M_{\Omega}$. Set $w_{\alpha} = w_{l, \Omega} w_{l, \Theta}$.*

1. $\mathbf{A}_N(\nu, \sigma, w_{\alpha})$ is holomorphic at $\nu = 0$ if and only if $\mathbf{A}_N(\nu, \hat{\sigma}, w_{\alpha})$ is holomorphic at $\nu = 0$.
2. Suppose that $w_{\alpha}\sigma \cong \sigma$. Then $A_N(\sigma, w_{\alpha})$ is a scalar if and only if $A_N(\hat{\sigma}, w_{\alpha})$ is a scalar.

Proof. 1. Let

$$\begin{aligned} [W_{\Theta} \setminus W] &= \{w \in W \mid w^{-1}\Theta > 0\}, \\ [W/W_{\Theta}] &= \{w \in W \mid w\Theta > 0\}. \end{aligned}$$

According to [C], page 7, the set $[W_{\Theta} \setminus W]$ (respectively, $[W/W_{\Theta}]$) is a set of representatives of cosets $W_{\Theta} \setminus W$ (respectively, W/W_{Θ}). Moreover, ([C], Lemma 1.1.2),

$$(1) \quad l(xw_1) = l(x) + l(w_1),$$

for any $x \in W_{\Theta}$, $w_1 \in [W_{\Theta} \setminus W]$,

$$(2) \quad l(w_2x) = l(w_2) + l(x),$$

for any $x \in W_{\Theta}$, $w_2 \in [W/W_{\Theta}]$.

Notice that $w_0 \in W_{\Theta}$ and $w_{\alpha}(\Theta) > 0$. Therefore, $w_{\alpha} \in [W/W_{\Theta}]$, $w_{\alpha}^{-1} \in [W_{\Theta} \setminus W]$. According to (1) and (2), we have

$$(3) \quad l(w_{\alpha}w_0) = l(w_{\alpha}) + l(w_0),$$

$$(4) \quad l(w_0w_{\alpha}^{-1}) = l(w_0) + l(w_{\alpha}^{-1}).$$

Let $w'_0 = w_{\alpha}w_0w_{\alpha}^{-1}$. Then

$$(5) \quad w_{\alpha}w_0 = w'_0w_{\alpha},$$

$$(6) \quad w_0 w_\alpha^{-1} = w_\alpha^{-1} w'_0.$$

Let $\Theta' = w_\alpha(\Theta)$. Then $w_\alpha^{-1}(\Theta') = \Theta > 0$, so $w_\alpha \in [W_{\Theta'} \setminus W]$, $w_\alpha^{-1} \in [W/W_{\Theta'}]$. Applying (1) and (2) to Θ' , w'_0 and w_α , we obtain

$$(7) \quad l(w'_0 w_\alpha) = l(w'_0) + l(w_\alpha),$$

$$(8) \quad l(w_\alpha^{-1} w'_0) = l(w_\alpha^{-1}) + l(w'_0).$$

It follows from (6) that

$$\mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma_0, w_0 w_\alpha^{-1}) = \mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma_0, w_\alpha^{-1} w'_0),$$

where $\nu \in \mathfrak{a}_{M, \mathbb{C}}^* \subseteq \mathfrak{a}_{M_0, \mathbb{C}}^*$. Now, (4) and (8) give

$$(9) \quad \mathbf{A}_N(\nu, \sigma_0, w_0) \mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma_0, w_\alpha^{-1}) = \mathbf{A}_N(w'_0 w_\alpha \nu, w'_0 w_\alpha \sigma_0, w_\alpha^{-1}) \mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma_0, w'_0).$$

According to (5), we have

$$\mathbf{A}_N(\nu, \sigma_0, w'_0 w_\alpha) = \mathbf{A}_N(\nu, \sigma_0, w_\alpha w_0),$$

which together with (3) and (7) give

$$(10) \quad \mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma_0, w'_0) \mathbf{A}_N(\nu, \sigma_0, w_\alpha) = \mathbf{A}_N(w_0 \nu, w_0 \sigma_0, w_\alpha) \mathbf{A}_N(\nu, \sigma_0, w_0).$$

We use the notation of [Sh2], pg.279, to express the following result of Harish-Chandra ([H-C]):

$$(11) \quad \mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma, w_\alpha^{-1}) \mathbf{A}_N(\nu, \sigma, w_\alpha) = \mu(w_\alpha \nu, w_\alpha \sigma, w_\alpha^{-1})^{-1} \gamma^2(N/P),$$

where $\mu(w_\alpha \nu, w_\alpha \sigma, w_\alpha)$ is the Plancherel measure and $\gamma(N/P)$ is a positive constant defined in [Sh2].

The representation σ_0 is irreducible supercuspidal. It follows from the Langlands classification that $\sigma_0 = \exp(\nu_0) \tau_0$, where $\nu_0 \in \mathfrak{a}_{M_0, \mathbb{C}}^*$ and τ_0 is tempered. Then $\exp(\nu) \sigma_0 = \exp(\nu + \nu_0) \tau_0$ and $\mathbf{A}_N(\nu, \sigma_0, w_\alpha) = \mathbf{A}_N(\nu + \nu_0, \tau_0, w_\alpha)$. As in (11), we have

$$\mathbf{A}_N(w_\alpha(\nu + \nu_0), w_\alpha \tau_0, w_\alpha^{-1}) \mathbf{A}_N(\nu + \nu_0, \tau_0, w_\alpha) = \mu(w_\alpha(\nu + \nu_0), w_\alpha \tau_0, w_\alpha^{-1})^{-1} \gamma^2(N/P_0),$$

so

$$(12) \quad \mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma_0, w_\alpha^{-1}) \mathbf{A}_N(\nu, \sigma_0, w_\alpha) = \mu(w_\alpha(\nu + \nu_0), w_\alpha \tau_0, w_\alpha^{-1})^{-1} \gamma^2(N/P_0).$$

Since σ is a subrepresentation of $i_{M, M_0}(\sigma_0)$, the operators $\mathbf{A}_N(\nu, \sigma, w_\alpha)$ and $\mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma, w_\alpha^{-1})$ are restrictions of $\mathbf{A}_N(\nu, \sigma_0, w_\alpha)$ and $\mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma_0, w_\alpha^{-1})$. The equality (12) tells us that, away from the poles, $\mathbf{A}_N(w_\alpha \nu, w_\alpha \sigma_0, w_\alpha^{-1}) \mathbf{A}_N(\nu, \sigma_0, w_\alpha)$

is a scalar. The same equality holds for any subspace of $i_{N,M_0}(V_0)$. In particular, it holds for $i_{N,M}(V) \hookrightarrow i_{N,M_0}(V_0)$, so (11) implies

$$(13) \quad \mathbf{A}_N(w_\alpha\nu, w_\alpha\sigma_0, w_\alpha^{-1})\mathbf{A}_N(\nu, \sigma_0, w_\alpha) = \mu(w_\alpha\nu, w_\alpha\sigma, w_\alpha^{-1})^{-1}\gamma^2(N/P).$$

Now, using (13), (9) and (10), we have

$$(14) \quad \begin{aligned} & \mu(w_\alpha\nu, w_\alpha\sigma, w_\alpha^{-1})^{-1}\gamma^2(N/P)\mathbf{A}_N(\nu, \sigma_0, w_0) \\ &= \mathbf{A}_N(\nu, \sigma_0, w_0)\mu(w_\alpha\nu, w_\alpha\sigma, w_\alpha^{-1})^{-1}\gamma^2(N/P) \\ &= \mathbf{A}_N(\nu, \sigma_0, w_0)\mathbf{A}_N(w_\alpha\nu, w_\alpha\sigma_0, w_\alpha^{-1})\mathbf{A}_N(\nu, \sigma_0, w_\alpha) \\ &= \mathbf{A}_N(w'_0w_\alpha\nu, w'_0w_\alpha\sigma_0, w_\alpha^{-1})\mathbf{A}_N(w_\alpha\nu, w_\alpha\sigma_0, w'_0)\mathbf{A}_N(\nu, \sigma_0, w_\alpha) \\ &= \mathbf{A}_N(w_\alpha w_0\nu, w_\alpha w_0\sigma_0, w_\alpha^{-1})\mathbf{A}_N(w_0\nu, w_0\sigma_0, w_\alpha)\mathbf{A}_N(\nu, \sigma_0, w_0). \end{aligned}$$

The equality (14) tells us that the restriction of $\mathbf{A}_N(w_\alpha w_0\nu, w_\alpha w_0\sigma_0, w_\alpha^{-1})\mathbf{A}_N(w_0\nu, w_0\sigma_0, w_\alpha)$ to the nonzero subspace $\text{im}(\mathbf{A}_N(\nu, \sigma_0, w_0))$ is equal to $\mu(w_\alpha\nu, w_\alpha\sigma, w_\alpha^{-1})^{-1}\gamma^2(N/P)$.

This implies

$$(15) \quad \mathbf{A}_N(w_\alpha w_0\nu, w_\alpha w_0\sigma_0, w_\alpha^{-1})\mathbf{A}_N(w_0\nu, w_0\sigma_0, w_\alpha) = \mu(w_\alpha\nu, w_\alpha\sigma, w_\alpha^{-1})^{-1}\gamma^2(N/P).$$

In particular, (15) holds on the subspace $i_{N,M}(\hat{V})$, so

$$(16) \quad \mathbf{A}_N(w_\alpha w_0\nu, w_\alpha \hat{\sigma}, w_\alpha^{-1})\mathbf{A}_N(w_0\nu, \hat{\sigma}, w_\alpha) = \mu(w_\alpha\nu, w_\alpha\sigma, w_\alpha^{-1})^{-1}\gamma^2(N/P).$$

The equalities (11) and (16) imply that $\mathbf{A}_N(\nu, \sigma, w_\alpha)$ is holomorphic at $\nu = 0$ if and only if $\mathbf{A}_N(\nu, \hat{\sigma}, w_\alpha)$ is holomorphic at $\nu = 0$. We will explain it in more detail.

The representation σ is square integrable and the operator $\mathbf{A}_N(\nu, \sigma, w_\alpha)$ is rank-one, so we may apply results from Section 5.4.2 of [S2].

If $w_\alpha\sigma \not\cong \sigma$, then $i_{N,M}(\sigma)$ is irreducible and $\mathbf{A}_N(\nu, \sigma, w_\alpha)$ is holomorphic at $\nu = 0$. Also ([S2], Corollary 5.4.2.2), $\mu(0, w_\alpha\sigma, w_\alpha^{-1}) > 0$. Corollary 3.4 tells us that $i_{N,M}(\hat{\sigma})$ is irreducible. From (16), we read that the composition of two standard intertwining operators on an irreducible space is a scalar. Since a standard intertwining operator is not zero, the operators in (16) have no poles at $w_0\nu = 0$.

Now, suppose that $w_\alpha\sigma \cong \sigma$. According to [S2], Corollary 5.4.2.3, $i_{N,M}(\sigma)$ is reducible if and only if $\mu(0, w_\alpha\sigma, w_\alpha^{-1}) > 0$. Notice that $w_\alpha^{-1} = w_\alpha$, so the operators $\mathbf{A}_N(0, \hat{\sigma}, w_\alpha)$ and $\mathbf{A}_N(0, w_\alpha\hat{\sigma}, w_\alpha^{-1})$ are equal (under the isomorphism $w_\alpha\sigma \cong \sigma$). If $\mu(0, w_\alpha\sigma, w_\alpha^{-1}) = 0$, the induced representations $i_{N,M}(\sigma)$ and $i_{N,M}(\hat{\sigma})$ are irreducible and the operators in (11) and (16) have poles at zero. If $\mu(0, w_\alpha\sigma, w_\alpha^{-1}) > 0$, the induced representations $i_{N,M}(\sigma)$ and $i_{N,M}(\hat{\sigma})$ are reducible, with

two inequivalent irreducible components. It follows that the operators in (11) and (16) are holomorphic at zero.

2. If $\mathbf{A}_N(\nu, \hat{\sigma}, w_\alpha)$ is holomorphic at $\nu = 0$, then $T_w \circ \mathbf{A}_N(\hat{\sigma}, w_\alpha)$ is non-scalar, so $A_N(\hat{\sigma}, w_\alpha)$ is non-scalar. \square

8. THE R -GROUP FOR $SO(m, F)$ AND $Sp(2m, F)$

In this section, G will denote a special orthogonal group or a symplectic group.

Let M be the standard Levi subgroup of G corresponding to $\Theta \subset \Delta$. Let σ be an irreducible square integrable representation of M and $\hat{\sigma}$ the Aubert involution of σ . Suppose that $\hat{\sigma}$ is unitary.

From Corollary 3.4, we have

$$\mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma)) \cong \mathrm{Hom}_G(i_{G,M}(\hat{\sigma}), i_{G,M}(\hat{\sigma})).$$

Let R denote the R -group for σ . Then, the set of normalized intertwining operators $\{A(\sigma, r) \mid r \in R\}$ is a basis for the commuting algebra $C(\sigma)$.

Theorem 8.1. *Let $G = SO(m, F)$ or $Sp(2m, F)$. Let M be a standard Levi subgroup of G . Suppose that σ is an irreducible square integrable representation of M such that its Aubert involution $\hat{\sigma}$ is unitary. Let R be the R -group for σ . Then $\hat{\sigma}$ has the same R -group as σ in the following sense:*

the set of normalized standard intertwining operators

$$\{A(\hat{\sigma}, r) \mid r \in R\}$$

is a basis for the commuting algebra

$$C(\hat{\sigma}) = \mathrm{Hom}_G(i_{G,M}(\hat{\sigma}), i_{G,M}(\hat{\sigma})).$$

Remark 8.1. *In the statement of Theorem 8.1, we make the assumption that σ is a square integrable representation such that its Aubert involution $\hat{\sigma}$ is unitary. It is conjectured that the Aubert involution of any unitary representation is unitary (in other words, that the Aubert involution preserves unitarity). This conjecture seems to be very difficult to prove. D. Barbasch and A. Moy in [BaM] proved the conjecture for representations which have nonzero Iwahori fixed vector, by using the Kazhdan-Lusztig parametrization of such representations.*

Proof. Set

$$S_k = \begin{cases} Sp(2k, F), \\ SO(2k+1, F), \\ SO(2k, F). \end{cases}$$

Then M is isomorphic to

$$M \cong GL(k_1, F) \times \cdots \times GL(k_n, F) \times S_k,$$

where $m = k_1 + \cdots + k_n + k$ ([T1], [B2]), and

$$\sigma = \rho_1 \otimes \cdots \otimes \rho_n \otimes \tau,$$

where ρ_i is a square integrable representation of $GL(k_i, F)$ and τ is a square integrable representation of S_k . We have

$$\hat{\sigma} = \hat{\rho}_1 \otimes \cdots \otimes \hat{\rho}_n \otimes \hat{\tau}.$$

The R -group for σ is computed in [G]. Let $s \in O(2m, F)$ be the sign change element (denoted by c_m in [G]) which induces the nontrivial automorphism on the Dynkin diagram. As in [G, §6], we consider for $G = SO(2m, F)$ four cases:

1. k_i is even, for $i = 1, \dots, n$.
2. $k > 0$ and, for $i = 1, \dots, n$, k_i is even or k_i is odd and $s\sigma \cong \sigma$.
3. $k = 0$ and k_i is odd, for $i = 1, \dots, n$.
4. $k > 0$ and k_i is odd, $s\sigma \not\cong \sigma$, for $i = 1, \dots, n$.

First, let

$$G = \begin{cases} Sp(2m, F), \\ SO(2m+1, F), \\ SO(2m, F), \end{cases} \quad M \text{ satisfies 1. or 2.}$$

Then ([B1])

$$W(\Theta) \cong Sym(n) \rtimes \{\pm 1\}^n.$$

For $1 \leq i \leq n$, define

$$\epsilon_i = (1, \dots, 1, -1, 1, \dots, 1),$$

where -1 is on the i -th place. Let $G_i = S_{k_i+k}$, $M_i = GL(k_i, F) \times S_k$. Set

$$S = \{\epsilon_i \mid i_{G_i, M_i}(\rho_i \otimes \tau) \text{ is reducible}\}.$$

It follows from [G], Lemma 6.3 and Theorems 6.4, 6.5, that the R -group for σ is a subgroup of the group generated by S . Let

$$\begin{aligned} i_{G, M}(V) &= V_1 \oplus \cdots \oplus V_l, \\ i_{G, M}(\hat{V}) &= \hat{V}_1 \oplus \cdots \oplus \hat{V}_l \end{aligned}$$

be the decomposition of $i_{G,M}(V)$, $i_{G,M}(\hat{V})$ as the direct sum of irreducible components. We know from [He], [G] that $i_{G,M}(\sigma)$ is multiplicity one. Then $i_{G,M}(\hat{\sigma})$ is multiplicity one and the correspondence $V_j \leftrightarrow \hat{V}_j$ is unique.

Take $\epsilon_i \in S$. Then $\rho_i \cong \tilde{\rho}_i$ ([T3], Lemma 2.1 and [B3], Lemma 4.3).

First, suppose that $i = n$. Denote by N the standard Levi subgroup of G generated by ϵ_i and M . Then M is a maximal Levi subgroup of N . The induced representations $i_{N,M}(\sigma)$ and $i_{N,M}(\hat{\sigma})$ are of the length two. Write

$$\begin{aligned} i_{N,M}(\sigma) &= W_1 \oplus W_2, \\ i_{N,M}(\hat{\sigma}) &= \hat{W}_1 \oplus \hat{W}_2, \end{aligned}$$

where W_1, W_2 are irreducible and $W_1 \not\cong W_2$. It follows from Lemma 5.2 that

$$\begin{aligned} A_N(\sigma, \epsilon_n)|_{W_1} &= 1, \\ A_N(\sigma, \epsilon_n)|_{W_2} &= -1, \end{aligned}$$

and

$$\begin{aligned} A(\sigma, \epsilon_n)|_{i_{N,M}(W_1)} &= 1, \\ A(\sigma, \epsilon_n)|_{i_{N,M}(W_2)} &= -1, \end{aligned}$$

(up to exchange of indices). According to Lemma 7.1, $A_N(\sigma, \epsilon_n)$ is non-scalar. Repeating the same arguments as in the proof of Lemma 5.2, we obtain

$$\begin{aligned} A_N(\hat{\sigma}, \epsilon_n)|_{\hat{W}_{j_1}} &= 1, \\ A_N(\hat{\sigma}, \epsilon_n)|_{\hat{W}_{j_2}} &= -1, \end{aligned}$$

and

$$\begin{aligned} A(\hat{\sigma}, \epsilon_n)|_{i_{N,M}(\hat{W}_{j_1})} &= 1, \\ A(\hat{\sigma}, \epsilon_n)|_{i_{N,M}(\hat{W}_{j_2})} &= -1, \end{aligned}$$

where $\{j_1, j_2\} = \{1, 2\}$. We may choose $c(\epsilon_n) = 1$ or -1 such that $c(\epsilon_n)A(\hat{\sigma}, \epsilon_n)$ acts on $\hat{V}_1 \oplus \cdots \oplus \hat{V}_l$ in the same way as $A(\sigma, \epsilon_n)$ acts on $V_1 \oplus \cdots \oplus V_l$.

Now, suppose that $i \neq n$. Let $p_{i,n}$ be the permutation on the set $\{1, \dots, n\}$ which interchanges i and n . Then $p_{i,n} = p_{i,n}^{-1}$ and $\epsilon_i = p_{i,n}\epsilon_n p_{i,n}$. We have

$$A'(\sigma, \epsilon_i) = A'(\epsilon_n p_{i,n} \sigma, p_{i,n}) A'(p_{i,n} \sigma, \epsilon_n) A'(\sigma, p_{i,n}).$$

Write

$$\begin{aligned} i_{G,M}(p_{i,n}\sigma) &= V'_1 \oplus \cdots \oplus V'_l, \\ i_{G,M}(p_{i,n}\hat{\sigma}) &= \hat{V}'_1 \oplus \cdots \oplus \hat{V}'_l, \end{aligned}$$

where $V'_j \cong V_j$, $\hat{V}'_j \cong \hat{V}_j$ for $1 \leq j \leq l$. As above, we may choose $c(\epsilon_i) = 1$ or -1 such that $c(\epsilon_i)A(p_{i,n}\hat{\sigma}, \epsilon_i)$ acts on $\hat{V}'_1 \oplus \cdots \oplus \hat{V}'_l$ in the same way as $A(p_{i,n}\sigma, \epsilon_n)$ acts on $V'_1 \oplus \cdots \oplus V'_l$. Take $j \in \{1, \dots, n\}$. Then $A(p_{i,n}\sigma, \epsilon_i)|_{V'_j} = 1$ or -1 , assume that $A(p_{i,n}\sigma, \epsilon_i)|_{V'_j} = -1$ (the proof for 1 goes in the same way). Then $c(\epsilon_i)A(p_{i,n}\hat{\sigma}, \epsilon_i)|_{\hat{V}'_j} = -1$. Moreover,

$$\begin{aligned} A(\sigma, \epsilon_i)|_{V_j} &= (A'(p_{i,n}\sigma, p_{i,n})(-1)A'(\sigma, p_{i,n}))|_{V_j} = -1, \\ c(\epsilon_i)A(\hat{\sigma}, \epsilon_i)|_{\hat{V}_j} &= (A'(p_{i,n}\hat{\sigma}, p_{i,n})(-1)A'(\hat{\sigma}, p_{i,n}))|_{\hat{V}_j} = -1. \end{aligned}$$

It follows that $c(\epsilon_i)A(\hat{\sigma}, \epsilon_i)$ acts on $\hat{V}_1 \oplus \cdots \oplus \hat{V}_l$ in the same way as $A(\sigma, \epsilon_i)$ acts on $V_1 \oplus \cdots \oplus V_l$.

Now, $\{A(\sigma, r) \mid r \in R\}$ is a basis for $C(\sigma)$ and $R \subseteq \langle S \rangle$. For $r \in R$, we write $r = s_1 \cdots s_k$, $s_1, \dots, s_k \in S$, and define $c(r) = c(s_1) \cdots c(s_k)$. Then $c(r) = 1$ or -1 and it does not depend on the choice of s_1, \dots, s_k . The above consideration shows that $\{c(r)A(\hat{\sigma}, r) \mid r \in R\}$ is a basis for $C(\hat{\sigma})$. This implies that $\{A(\hat{\sigma}, r) \mid r \in R\}$ is a basis for $C(\hat{\sigma})$.

Now, suppose that $G = SO(2m, F)$ and that M satisfies 3. or 4. Then, by [G], R is a subgroup of the group generated by

$$\{\epsilon_i \epsilon_j \mid \tilde{\rho}_i \cong \rho_i, \tilde{\rho}_j \cong \rho_j, \rho_i \not\cong \rho_j\}.$$

We consider the case

$$\begin{aligned} M &\cong GL(k_1) \times GL(k_2) \times SO(2k), \\ \sigma &= \rho_1 \otimes \rho_2 \otimes \tau, \end{aligned}$$

$k_1 + k_2 + k = m$. The general case follows in the same way as earlier.

Suppose that M satisfies 3. Then k_1, k_2 are odd, $k = 0$. Suppose that $k_1, k_2 > 1$. The simple roots are $\alpha_i = e_i - e_{i+1}$, $1 \leq i \leq m-1$, $\alpha_m = e_{m-1} + e_m$ ([B2], [G]). The set of simple roots corresponding to M is $\Theta = \rho \setminus \{\alpha_{k_1}, \alpha_m\}$. We can write $\epsilon_1 \epsilon_2 = (-1, -1)$ as in Lemma 6.1 in the following way

$$\epsilon_1 \epsilon_2 = w_4 w_3 w_2 w_1,$$

where

$$\begin{aligned} \Theta_1 &= \Theta, & \Omega_1 &= \rho \setminus \{\alpha_{k_1}\}, \\ \Theta_2 &= \rho \setminus \{\alpha_{k_1}, \alpha_{m-1}\}, & \Omega_2 &= \rho \setminus \{\alpha_{m-1}\}, \\ \Theta_3 &= \rho \setminus \{\alpha_{k_2}, \alpha_{m-1}\}, & \Omega_3 &= \rho \setminus \{\alpha_{k_2}\}, \\ \Theta_4 &= \rho \setminus \{\alpha_{k_2}, \alpha_m\}, & \Omega_4 &= \rho \setminus \{\alpha_m\}, \\ \Theta_5 &= \Theta. \end{aligned}$$

Then, by Corollary 6.3,

$$\begin{aligned} \mathbf{A}(\nu, \sigma, \epsilon_1 \epsilon_2) &= \mathbf{A}(\nu_4, \sigma_4, w_4) \cdots \mathbf{A}(\nu_1, \sigma_1, w_1), \\ \mathbf{A}(\nu, \hat{\sigma}, \epsilon_1 \epsilon_2) &= \mathbf{A}(\nu_4, \hat{\sigma}_4, w_4) \cdots \mathbf{A}(\nu_1, \hat{\sigma}_1, w_1). \end{aligned}$$

According to Lemma 5.1, $\mathbf{A}(\nu_i, \sigma_i, w_i)$ is holomorphic at $\nu_i = 0$, for $1 \leq i \leq 4$. Lemma 7.1 tells us that $\mathbf{A}(\nu_i, \hat{\sigma}_i, w_i)$ is holomorphic at $\nu_i = 0$, for $1 \leq i \leq 4$. Therefore, $\mathbf{A}(\nu, \sigma, \epsilon_1 \epsilon_2)$ and $\mathbf{A}(\nu, \hat{\sigma}, \epsilon_1 \epsilon_2)$ are holomorphic at $\nu = 0$. It follows from [G], Theorem 6.8, that $i_{G,M}(\rho_1 \otimes \rho_2)$ has the length two. By Corollary 3.4, $i_{G,M}(\hat{\rho}_1 \otimes \hat{\rho}_2)$ also has the length two. Now we can apply Lemma 5.2 to prove that $A(\sigma, \epsilon_1 \epsilon_2)$ acts on

$$i_{G,M}(\rho_1 \otimes \rho_2) = V_1 \oplus V_2$$

in the same way as $A(\hat{\sigma}, \epsilon_1 \epsilon_2)$ acts on

$$i_{G,M}(\hat{\rho}_1 \otimes \hat{\rho}_2) = \hat{V}_1 \oplus \hat{V}_2$$

(up to ± 1).

The other cases (when $k \neq 0$, $k_1 = 1$ or $k_2 = 1$) can be proved in a similar manner. \square

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