THE LANGLANDS CLASSIFICATION FOR NON-CONNECTED p-ADIC GROUPS

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Abstract. We give the Langlands classification for a non connected reductive quasi-split p-adic group G, under the assumption that G/G^0 is abelian (here G^0 denotes the connected component of the identity of G). The Langlands classification for non-connected groups is an extension of the Langlands classification from the connected case.

1. Introduction

Suppose G is the F-points of a connected, reductive group defined over a nonarchimedean local field F. The Langlands classification (cf. [S], [B-W]) gives a bijective correspondence

$$Irr(G) \longleftrightarrow Lang(G)$$

between irreducible, admissible representations of G and triples of Langlands data. In this paper, we extend the Langlands classification to cover the case where G is the F-points of a non-connected, quasi-split, reductive group over F, subject to the condition G/G^0 is finite and abelian (finiteness being automatic, cf. 7.3 [Hu]). The result is given as Theorem 4.2.

The Langlands classification was originally done in the context of connected real groups (cf. [L]). This has been extended to non-connected real groups in [M]. In [M], the Langlands classification is essentially reproven under weaker hypotheses. Our approach is different–our results do not contain the connected case as a special case, but rather, they use the connected case as a starting point. In particular, following the lead of [Go], [Go-H], we use a lemma of [G-K] to move from the connected case to the non-connected case.

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Let us now give a rough idea of how the proof goes. For simplicity, let us assume G/G^0 has prime order and $G = G^0 \rtimes C$, C the component group. We take representatives for C which stabilize the Borel subgroup of G^0. Let π be an irreducible admissible representation of G; π_0 an irreducible (admissible) representation of G^0 with $\pi_0 \subset \mathrm{Res}^G_{G^0}(\pi)$. We shall describe how to use the Langlands data for π_0 to obtain Langlands data for π. Write $\pi_0 = L(P, \nu, \tau)$, where (P, ν, τ) is the Langlands data for π_0. We note that $P = MU$ is a parabolic subgroup of G, $\nu \in \mathfrak{a}^*_-$ ($\mathfrak{a} = \mathrm{Lie}(A)$, where A is maximal split torus in the center of M), and τ is an irreducible, tempered representation of M. It is worth noting that we work in the subrepresentation setting of the Langlands classification, hence the presence of $\mathfrak{a}^*_-$ rather than $\mathfrak{a}^*_+$. (See Remark 4.2 for a discussion of the Langlands classification in the quotient setting.) For this discussion, let us write M^0 instead of M, thereby freeing us to use M for its (possibly non-connected) counterpart in G.

At this point, there are two possibilities: either $\mathrm{Res}^G_{G^0}(\pi)$ is reducible or irreducible. First, let us suppose it is reducible. By a corollary of a lemma from [G-K], given as Lemma 2.1 in this paper, we have

$$\mathrm{Res}^G_{G^0}(\pi) = \bigoplus_{c \in C} c \cdot \pi_0$$

and

$$\pi \cong \mathrm{Ind}^G_{G^0}(c \cdot \pi_0)$$

for all $c \in C$. Further, we note that

$$c \cdot \pi_0 = L(c \cdot P, c \cdot \nu, c \cdot \tau)$$

for all $c \in C$ (cf. Proposition 4.5). Now, suppose $c \cdot P \neq P$ when $c \neq 1$. In this case, our definition of Langlands data–and more precisely, our definition of standard parabolic subgroups–singles out one parabolic subgroup from $\{c \cdot P\}$ (these are non-conjugate in G^0 but conjugate in G), call it $c_0 \cdot P$. The Langlands data for π is then $c_0 \cdot (P, \nu, \tau)$. Next, suppose $c \cdot P = P$ for all $c \in C$, but $c \cdot \nu \neq \nu$ when $c \neq 1$. In this case, C acts on $\mathfrak{a}^*$, subdividing $\mathfrak{a}^*_-$ into subchambers. Our definition of Langlands data singles out one of these as negative. Then, there is a $c_0 \in C$ such that $c_0 \cdot \nu$ lies in this subchamber. The Langlands data for π is then $c_0 \cdot (P, \nu, \tau)$. Finally, suppose $c \cdot P = P$,

$c \cdot \nu = \nu$ for all $c \in C$, but $c \cdot \tau \not\cong \tau$. In this case, it follows from the Lemma 2.1 that

$$\mathrm{Ind}_{M^0}^{M}(\tau) \cong \tau',$$

with τ' an irreducible representation of $M = M^0 \rtimes C$. In this case, our Langlands data for π is $(P \rtimes C, \nu, \tau')$. We note that it follows from the Lemma 2.1 and Proposition 4.5 that these three possibilities are the only possibilities which have $\mathrm{Res}_{G^0}^{G}(\pi)$ reducible.

Now, we consider the case where $\mathrm{Res}_{G^0}^{G}(\pi)$ is irreducible. In this case,

$$\mathrm{Res}_{G^0}^{G}(\pi) = \pi_0$$

and

$$\mathrm{Ind}_{G^0}^{G}(\pi_0) = \bigoplus_{\chi \in \widehat{G/G^0}} \chi \cdot \pi$$

where $\widehat{G/G^0}$ consists of characters of G which are trivial on G^0. For this to be the case, we must have $c \cdot P = P$, $c \cdot \nu = \nu$, and $c \cdot \tau \cong \tau$ for all $c \in C$ (by Lemma 2.1 and Proposition 4.5). In this case, it follows that

$$\mathrm{Ind}_{M^0}^{M}(\tau) = \bigoplus_{\chi \in \widehat{M/M^0}} \chi \cdot \tau',$$

where τ' is an irreducible component of $\mathrm{Ind}_{M^0}^{M}(\tau)$. The Langlands data for π is then $(P \rtimes C, \nu, \chi\tau')$ for an appropriately chosen $\chi\tau'$.

To deal with the case when G/G^0 has order which is not prime, we take a filtration

$$G^0 = G_1 \subset G_2 \subset \cdots \subset G_k = G$$

where G_i/G_{i-1} has prime order. We can then use the above argument in an inductive fashion. Not surprisingly, the starting point for this inductive argument is the Langlands classification in the case of connected groups.

We now discuss the results section by section. In the next section, we give notation and preliminaries. The third section discusses the parabolic subgroups needed. In the fourth section, we give the statement and proof of the Langlands classification for the non-connected groups under consideration.

We make one additional remark. Part of our interest in this project is its applications to $O(2n, F)$. Because of certain structural similarities to $Sp(2n, F)$ and $SO(2n+1, F)$ (e.g., compare [T1] and [B]), many results which hold for these latter two families of classical groups should also hold for $O(2n, F)$. But, in order to capitalize on these similarities, some of the machinery which holds for connected groups must be extended to the non-connected case. One such piece of machinery is the Langlands classification, the subject of this paper. We note that in order to make it as easy as possible to extend results such as [T2], [J1], etc., from $Sp(2n, F)$, $SO(2n+1, F)$ to $O(2n, F)$, it is advantageous to have the Langlands classification for $O(2n, F)$ in a form similar to that for $Sp(2n, F)$, $SO(2n+1, F)$. Therefore, at certain points in this paper where we need to make arbitrary choices (e.g., X_C and $\mathfrak{a}^*_-(C)$ in section 3), we make the choice which makes it easiest to get this similar form.

A brief discussion of the hypotheses is in order. Our results are based largely on the Langlands classification in the connected case ([B-W], [S]; our point-of-view is closer to [S]) and the results in section 2 [G-K]. Thus, since [B-W], [S] work in the context of reductive groups over nonarchimedean local fields and section 2 [G-K] works in the context of totally disconnected groups, we can allow $char F \neq 0$. On the other hand, while neither of these requires the group to be quasi-split, it is a convenient assumption for us: we then have a Borel subgroup and can choose representatives for G/G^0 which fix the Borel subgroup (under conjugation), act on the simple roots, etc.

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2. Notation and preliminaries

In this section, we introduce notation and give some background results. In particular, the main technical lemma (Lemma 2.1; a consequence of [G-K]) and the definition of tempered we need (Definition 2.5) are given in this section.

Let F be a p-adic field and G the group of F-points of a quasi-split reductive algebraic group defined over F. Let G^0 denote the connected component of the identity in G. We shall assume that

$$C = G/G^0$$

is a finite abelian group.

In the group G^0, fix a Borel subgroup $P_\emptyset \subset G^0$ and a maximal split torus $A_\emptyset \subset P_\emptyset$. We let Π denote the corresponding set of simple roots. For $\Phi \subset \Pi$, we let $P_\Phi = M_\Phi U_\Phi$ denote the standard parabolic subgroup determined by Φ.

Let $P = MU \subset G^0$ be a standard parabolic subgroup of G^0, A the split component of the center of M, $X(M)_F$ the group of F-rational characters of M. Let

$$\mathfrak{a} = \mathrm{Hom}(X(M)_F, \mathbb{R}) = \mathrm{Hom}(X(A)_F, \mathbb{R})$$

be the real Lie algebra of A and

$$\mathfrak{a}^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} = X(A)_F \otimes_{\mathbb{Z}} \mathbb{R}$$

its dual. There is a homomorphism (cf. [H]) $H_M\colon M \to \mathfrak{a}$ such that

$$q^{\langle \chi, H_M(m)\rangle} = |\chi(m)|$$

for all $m \in M$, $\chi \in X(M)_F$. Given $\nu \in \mathfrak{a}^*$, let us write

$$exp\,\nu = q^{\langle \nu, H_M(\cdot)\rangle}$$

for the corresponding character.

Before we go into notation and basic definitions for G, we need to do a couple of things. First, we fix a choice of representatives for G/G^0 which stabilize the Borel subgroup, hence act on the simple roots. By abuse of notation, we use C for both the component group G/G^0 and this set of representatives. Also, we want the inner product on $\mathfrak{a}_0^*$ to be C-invariant. If the standard inner product $\langle\, ,\rangle_0\colon \mathfrak{a}_0^* \times \mathfrak{a}_0^* \to \mathbb{R}$ is not, we can replace it with $\sum_{c\in C} c \cdot \langle\, ,\rangle_0$, where $c \cdot \langle x, y\rangle_0 = \langle c\cdot x, c \cdot y\rangle_0$ for $x, y \in \mathfrak{a}_0^*$.

Suppose that G_1, G_2 are subgroups of G, $G^0 \leq G_1 \leq G_2 \leq G$. We will use the notation i_{G_2,G_1} and r_{G_1,G_2} for induction and restriction of representations: If (σ, V) is an admissible representation of G_1, then $i_{G_2,G_1}(\sigma)$ is the representation of G_2 given by right translations on the space

$$V' = \{f\colon G_2 \to V \mid f(g_1 g) = \sigma(g_1) f(g),\ g \in G_2,\ g_1 \in G_1\}.$$

For an admissible representation π of G_2, $r_{G_1,G_2}(\pi) = \pi|_{G_1}$.

Set $D = G_2/G_1$. If π_1 is an irreducible representation of G_1 and $d \in D$, we define $d \cdot \pi_1$ by

$$d \cdot \pi_1(g) = \pi_1((d')^{-1} g d'),$$

for all $g \in G_1$, where $d' \in G_2$ is a representative of d. The equivalence class of $d \cdot \pi_1$ does not depend on choice of a representative d'.

Let $\hat{D}$ denote the set of all characters of D. If $\chi \in \hat{D}$, then we can think of χ as a character of G_2 which is trivial on G_1. According to [G-K] and [Go-H], we have the following properties of $r_{G_1,G_2}(\pi_2)$.

Lemma 2.1. *Suppose that $D = G_2/G_1$ is of prime order. Then for any irreducible admissible representation π_2 of G_2, the representation $r_{G_1,G_2}(\pi_2)$ is of multiplicity one. If π_1 is an irreducible component of $r_{G_1,G_2}(\pi_2)$, then π is an admissible representation of G_1 and either*

1. $$r_{G_1,G_2}(\pi_2) = \pi_1, \quad i_{G_2,G_1}(\pi_1) \cong \bigoplus_{\chi \in \hat{D}} \chi \otimes \pi_2,$$

   with $\{\chi \otimes \pi_2\}_{\chi \in \hat{D}}$ pairwise inequivalent, or

2. $$r_{G_1,G_2}(\pi_2) = \bigoplus_{d \in D} d \cdot \pi_1, \quad i_{G_2,G_1}(d \cdot \pi_1) \cong \pi_2, \forall d \in D,$$

   where $\{d \cdot \pi_1\}_{d \in D}$ are pairwise inequivalent.

Proof. These claims follow from the results in section 2 [G-K] in essentially the same way that Lemmas 2.12 and 2.13 [Go-H] do. (The main difference here is that we are not assuming $G_1 = G^0$.) □

To define supercuspidality (square-integrability, temperedness) of an irreducible representation π of G, we will look at the components of $r_{G^0,G}(\pi)$. In order for our definitions to make sense, we first need to establish certain properties about Haar measure.

Let μ denote a Haar measure on G^0. For $c \in C$, we define

$$(\mu \circ c)(S) = \mu(cSc^{-1}),$$

for $S \subset G^0$ a measurable set.

Lemma 2.2. *$\mu \circ c = \mu$.*

Proof. First, since $\mu \circ c$ is invariant under left translations, it is a Haar measure on G^0. Therefore, there exists a positive real number λ such that $\mu \circ c = \lambda\mu$. To see that $\lambda = 1$, let K be a compact open subgroup of G^0. Then

$$K_1 = \bigcap_{d \in C} (dKd^{-1})$$

is an open compact subgroup of G^0, and $(\mu \circ c)(K_1) = \mu(K_1)$. It follows that $\lambda = 1$. □

Let Z denote the center of G^0, $c \in C$. For $z \in Z$, an easy check tells us $czc^{-1} \in Z$. Hence, $cZc^{-1} = Z$. If $\bar{\mu}$ denotes the quotient measure on G^0/Z, we may now define $\bar{\mu} \circ c$ as above.

Corollary 2.3. $\bar{\mu} \circ c = \bar{\mu}$.

□

Proposition 2.4. *Let π be an irreducible representation of G^0, $c \in C$. Then π is square-integrable (resp., tempered) if and only if $c \cdot \pi$ is square-integrable (resp., tempered).*

Proof. Let V be the space of representation π, $v \in V$, $\tilde{v} \in \tilde{V}$. Let $f_{v,\tilde{v}}$ denote the matrix coefficient of π associated to v and $\tilde{v}$:

$$f_{v,\tilde{v}}(g) = \langle \pi(g)v, \tilde{v} \rangle$$

for all $g \in G^0$ (cf. [C]). Then $c \cdot f_{v,\tilde{v}}$, defined by $c \cdot f_{v,\tilde{v}}(g) = f_{v,\tilde{v}}(c^{-1}gc)$ for all $g \in G^0$, is a matrix coefficient of $c \cdot \pi$. By the preceding corollary,

$$\int_{G^0/Z} |c \cdot f_{v,\tilde{v}}|^2 d\bar{\mu} = \int_{G^0/Z} |f_{v,\tilde{v}}|^2 d(\bar{\mu} \circ c) = \int_{G^0/Z} |f_{v,\tilde{v}}|^2 d\bar{\mu}.$$

Thus $c \cdot f_{v,\tilde{v}} \in L^2(G^0/Z)$ if and only if $f_{v,\tilde{v}} \in L^2(G^0/Z)$. For temperedness, we just replace L^2 by $L^{2+\varepsilon}$. □

The following is now well-defined:

Definition 2.5. *Let π be an irreducible admissible representation of G. Let us call π* **supercuspidal** *(resp.,* **square-integrable**, **tempered***) if the components of $r_{G^0,G}(\pi)$ are supercuspidal (resp., square-integrable, tempered).*

Remark 2.1. *In terms of matrix coefficients, we have essentially defined π to be supercuspidal (resp., square-integrable, tempered) if its matrix coefficients are compactly supported in G/Z (resp., lie in $L^2(G/Z)$, $L^{2+\varepsilon}(G/Z)$), where Z is the center of G^0. This avoids certain problems which arise if the centers of G and G^0 are different (cf. [Go-H]).*

3. Parabolic subgroups

In this section, we introduce the standard parabolic subgroups of G which will be needed later (Definition 3.1). We also introduce the subchamber $\mathfrak{a}_-^*(C)$ in the dual of the Lie algebra of A, which plays the same role in the Langlands classification for G that $\mathfrak{a}_-^*$ plays in the Langlands classification for G^0.

Let $P_\Phi = M_\Phi U_\Phi \subset G^0$ be the standard parabolic subgroup of G^0 corresponding to $\Phi \subset \Pi$. Let

$$C(\Phi) = \{c \in C \mid c \cdot \Phi = \Phi\}.$$

We let

$$M_{\Phi,C(\Phi)} = \langle M_\Phi, C(\Phi)\rangle,$$

noting that our earlier abuse of notation allows us to interpret $C(\Phi)$ as a subset of the representatives for C. More generally, if $D \subset C(\Phi)$, we let

$$M_{\Phi,D} = \langle M_\Phi, D\rangle$$

(note that $M_{\Phi,1} = M_\Phi$). Suppose that M satisfies

$$M_\Phi \leq M \leq M_{\Phi,C(\Phi)}$$

(such an M has the form $M_{\Phi,D}$). We will consider subgroups of the form $P = MU = M_{\Phi,D}U_\Phi$. We write $P_{\Phi,D} = M_{\Phi,D}U_\Phi$. Since M normalizes U, we can define functors $i_{G,M}$ and $r_{M,G}$ as in [B-Z]. If (σ, W) is a smooth representation of M, we define the induced representation $i_{G,M}(\sigma)$ as follows: The group G acts on

$$\{f\colon G \to W \mid f(umg) = \delta_P^{1/2}(m)\sigma(m)f(g),\ u \in U,\ m \in M,\ g \in G\}$$

by the right translations

$$(R_g f)(x) = f(xg), \quad x, g \in G;$$

the smooth part of this representation is $i_{G,M}(\sigma)$. If (π, V) is a smooth representation of G, the representation $r_{M,G}(\pi)$ acts on the space

$$V_U = V/V(U),$$

where

$$V(U) = span_{\mathbb{C}}\{\pi(u)v - v | \, u \in U, \, v \in V\}.$$

The action of $m \in M$ on V_U is given by

$$r_{M,G}(\pi)(m)[v + V(U)] = \delta_P^{-1/2}(m)[\pi(m)v + V(U)].$$

The standard properties of the functors $i_{G,M}$ and $r_{M,G}$ are described in Proposition 1.9. [B-Z].

In G^0, the standard parabolic subgroups are non-conjugate. We would like to arrange this for G as well. Observe that for any $c \in C$, we have

$$cM_\Phi c^{-1} = M_{c\cdot\Phi},$$
$$cC(\Phi)c^{-1} = C(c \cdot \Phi),$$

so the groups $M_{\Phi,C(\Phi)}$ and $M_{c\cdot\Phi,C(c\cdot\Phi)}$ are conjugate. Similarly, if $M_\Phi \leq M \leq M_{\Phi,C(\Phi)}$, then $M = M_{\Phi,D}$, where $D \leq C(\Phi)$, and

$$cMc^{-1} = M_{c\cdot\Phi,cDc^{-1}} \leq M_{c\cdot\Phi,C(c\cdot\Phi)}.$$

To arrange standard parabolic subgroups for G to be non-conjugate, we need to choose one group from among $\{M_{c\cdot\Phi}\}_{c\in C}$, i.e., a representative of the set $\{c \cdot \Phi\}_{c\in C}$.

Choose an ordering on the elements of Π. Then, one has a lexicographic order on the subsets of Π. (To be precise, if $\Phi_1 = \{\beta_1, \dots, \beta_k\}$ and $\Phi_2 = \{\gamma_1, \dots, \gamma_l\}$ with $\beta_1 > \cdots > \beta_k$ and $\gamma_1 > \cdots > \gamma_l$, we write $\Phi_1 \succ \Phi_2$ if $\beta_1 > \gamma_1$ or $\beta_1 = \gamma_1$ and $\beta_2 > \gamma_2$, etc. The absence of a root is lower than a root, so $\emptyset$ is minimal.) We define

$$X_C = \{\Phi \subset \Pi | \; \Phi \text{ is maximal among } \{c \cdot \Phi\}_{c\in C}\}.$$

In particular, any $\Phi \subset \Pi$ is conjugate in G to an element of X_C.

Definition 3.1. *Let $P_\Phi = M_\Phi U_\Phi \subset G^0$ be the standard parabolic subgroup of G^0 corresponding to $\Phi \subset \Pi$. We call $P = MU_\Phi$, where*

$$M_\Phi \leq M \leq M_{\Phi,C(\Phi)}$$

and $\Phi \in X_C$, a **standard parabolic subgroup** *of G.*

Let $P = MU$ be a standard parabolic subgroup of G. Write $P^0 = P \cap G^0 = M_\Phi U_\Phi$ and $M = M_{\Phi,D}$. We denote the split component of M_Φ by A. Let $\mathfrak{a}$ be the real Lie algebra of A, and $\mathfrak{a}^*$ its dual. Let $\Pi(P^0, A) \subset \mathfrak{a}^*$ denote the set of simple roots corresponding to the pair (P^0, A); these are the nonzero projections to $\mathfrak{a}^*$ of elements of Π.

If we identify $\mathfrak{a}^*$ with a subspace of $\mathfrak{a}_0^*$, we get a $C(\Phi)$-invariant inner product $\langle\, ,\rangle\colon \mathfrak{a}^* \times \mathfrak{a}^* \to \mathbb{R}$. As in [S], we set

$$\mathfrak{a}_-^* = \{x \in \mathfrak{a}^* |\ \langle x, \alpha\rangle < 0,\ \forall \alpha \in \Pi(P^0, A)\}.$$

While elements of $\mathfrak{a}_-^*$ are not conjugate in G^0, they may be conjugate in G. Therefore, we choose a subchamber of $\mathfrak{a}_-^*$:

$$\mathfrak{a}_-^*(C) = \{x \in \mathfrak{a}_-^* |\ x \succeq c \cdot x,\ \forall c \in C(\Phi)\}.$$

Here $\prec$ is the lexicographic order inherited from the order on Π: Write $\Pi(P^0, A) = \{\alpha_1, \dots, \alpha_j\}$, where $\alpha_1 > \cdots > \alpha_j$ with respect to the order on Π. Then $x, x' \in \mathfrak{a}^*$ has $x \prec x'$ if $\langle x, \alpha_1\rangle < \langle x', \alpha_1\rangle$ or $\langle x, \alpha_1\rangle = \langle x', \alpha_1\rangle$ and $\langle x, \alpha_2\rangle < \langle x', \alpha_2\rangle$, etc. (If x and x' are equal with respect to $\prec$, then $x - x'$ is perpendicular to all the roots in $\Pi(P^0, A)$.) We note that $\mathfrak{a}_-^*(C)$ is convex.

4. Langlands classification

In this section, we give the statement and proof of the Langlands classification for G (Theorem 4.2). Also of some potential interest is Proposition 4.5, which essentially deals with the effects of the action of C on the Langlands data of representations G^0.

Let

$$Irr(G)$$

denote the set of equivalence classes of all admissible irreducible representations of G. If π is an irreducible admissible representation of G, we write $\pi \in Irr(G)$, identifying π with its equivalence class.

Definition 4.1. *A* **set of Langlands data** *for G is a triple (P, ν, τ) with the following properties:*

1. *$P = MU$ is a standard parabolic subgroup of G.*
2. *$\nu \in \mathfrak{a}_-^*(C)$.*
3. *$M = M_{\Phi, C(\Phi,\nu)}$, where $C(\Phi, \nu) = \{c \in C(\Phi) |\ c \cdot \nu = \nu\}$.*
4. *$\tau \in Irr(M)$ is tempered.*

The Langlands classification will be a subrepresentation form of [S] (extended to non-connected groups). In particular, we are taking ν to be real rather than complex. (This subrepresentation variation of the Langlands classification for connected p-adic groups is described in [J2], e.g.)

For $\nu \in \mathfrak{a}^*_-(C)$, let $exp\,\nu$ be the character of M_Φ defined in Section 2. For $M = M_{\Phi,C(\Phi,\nu)}$, we extend $exp\,\nu$ to M by setting

$$exp\,\nu(mc) = exp\,\nu(m),$$

$m \in M_\Phi$, $c \in C(\Phi,\nu)$. Since

$$\begin{aligned} exp\,\nu(m_1c_1m_2c_2) &= exp\,\nu(m_1c_1m_2c_1^{-1}c_1c_2) \\ &= exp\,\nu(m_1)exp(c_1\cdot\nu)(m_2) \\ &= exp\,\nu(m_1c_1)exp\,\nu(m_2c_2), \end{aligned}$$

$exp\,\nu$ is a character of M. Here we use Condition 3. from the definition of Langlands data. This condition may be seen in Definition 5.6. [M], and it ensures that $exp\,\nu$ is a character of M.

If τ is a representation of M, then $exp\,\nu \otimes \tau$ is the representation of M defined by

$$(exp\,\nu \otimes \tau)(mc) = exp\,\nu(m)\tau(mc).$$

Theorem 4.2. (Langlands classification)
There is a bijective correspondence

$$Lang(G) \longleftrightarrow Irr(G),$$

where $Lang(G)$ denotes the set of all triples of Langlands data. Further, if $(P,\nu,\tau) \leftrightarrow \pi$ under this correspondence, then π is the unique irreducible subrepresentation of $i_{G,M}(exp\,\nu \otimes \tau)$.

If $(P,\nu,\tau) \leftrightarrow \pi$, then we write $\pi = L(P,\nu,\tau)$.

The basic idea of the proof is as follows: suppose

$$G^0 = G_0 \subset G_1 \subset \cdots \subset G_k = G$$

has $|G_i/G_{i-1}|$ prime for $i = 1,\dots,k$. We argue inductively, assuming the Langlands classification holds for G_{i-1} and showing that it holds for G_i. Starting the induction, of course, is the Langlands classification for connected groups.

For convenience, let $G_1 \subset G_2$ be two consecutive groups in the filtration above (not necessarily the first two). Then $G_1/G^0 = C_1$ and $G_2/G^0 = C_2$ with $C_1 \subset C_2 \subset C$ and $|C_2/C_1|$ prime.

Lemma 4.3. *Let $(P_1, \nu_1, \tau_1) \in Lang(G_1)$. Write $P_1^0 = P_{\Phi_1}$. For $i = 1, 2$, let*

$$C_i(\Phi_1) = \{c \in C_i | \; c \cdot \Phi_1 = \Phi_1\},$$
$$C_i(\Phi_1, \nu_1) = \{c \in C_i | \; c \cdot \Phi_1 = \Phi_1 \text{ and } c \cdot \nu_1 = \nu_1\},$$
$$C_i(\Phi_1, \nu_1, \tau_1) = \{c \in C_i | \; c \cdot \Phi_1 = \Phi_1, c \cdot \nu_1 = \nu_1 \text{ and } c \cdot \tau_1 = \tau_1\}.$$

We have either

1. $C_2(\cdot) = C_1(\cdot)$,

or 2. $C_2(\cdot)/C_1(\cdot) \cong C_2/C_1$,

where $(\cdot)$ denotes any of these (i.e., (Φ_1), (Φ_1, ν_1) or (Φ_1, ν_1, τ_1)).

Proof. Observe that $C_1(\cdot) = C_2(\cdot) \cap C_1$. If $C_2(\cdot) \subset C_1$, then $C_2(\cdot)/C_1(\cdot) = 1$. Suppose $C_2(\cdot) \not\subset C_1$. Then, $C_2(\cdot)/C_1(\cdot) \neq 1$. Choose representatives $c_2^{(1)}, \ldots, c_2^{(k)}$ for $C_2(\cdot)/C_1(\cdot)$. We claim that for $i \neq j$, $c_2^{(i)} C_1 \neq c_2^{(j)} C_1$. If this were the case, we would have $c_2^{(j)-1} c_2^{(i)} \in C_1$. Since $c_2^{(j)-1} c_2^{(i)} \in C_2(\cdot)$, we get $c_2^{(j)-1} c_2^{(i)} \in C_1 \cap C_2(\cdot) = C_1(\cdot)$, a contradiction. □

Lemma 4.4. *Let $P_1 = M_1 U_1$ be a standard parabolic subgroup of G_1, τ_1 a representation of M_1. For $c \in C_2$,*

$$c \cdot i_{G_1, M_1}(\tau_1) \cong i_{G_1, c \cdot M_1}(c \cdot \tau_1),$$

where the parabolic induction in the right-hand side is with respect to $c \cdot P_1 = c P_1 c^{-1}$.

Proof. Straightforward. Note that $c \cdot P_1$ is not necessarily standard. □

Suppose (P_1, ν_1, τ_1) is a set of Langlands data for G_1. Write $P_1^0 = P_\Phi = P$. For $d \in C_2/C_1$, we shall choose a representative $d(\Phi, \nu_1) \in dC_1$. Let

$$[d(\Phi)] = \{d' \in dC_1 | \; d' \cdot \Phi \text{ is maximal among } \{dc_1 \cdot \Phi\}_{c_1 \in C_1}\}.$$

If d' is an element of $[d(\Phi)]$, then $[d(\Phi)] = d' C_1(\Phi)$. Set

$$\Phi' = d' \cdot \Phi.$$

Note that this does not depend on choice of d' in $[d(\Phi)]$. We have $\mathfrak{a}'^* = d' \cdot \mathfrak{a}^*$, $d' \cdot \Pi(P, A) = \Pi(P', A')$. Since $\langle \, , \rangle$ is C-invariant, it follows that $d' \cdot \mathfrak{a}^*_- = \mathfrak{a}'^*_-$. Now, let

$$[d(\Phi, \nu_1)] = \{d' \in [d(\Phi)] | \, d' \cdot \nu_1 \succeq d'' \cdot \nu_1, \, \forall d'' \in [d(\Phi)]\}.$$

If d'' is an element of $[d(\Phi, \nu_1)]$, then $[d(\Phi, \nu_1)] = d''C_1(\Phi, \nu_1)$. We choose an element of $[d(\Phi, \nu_1)]$ and denote it by

$$d(\Phi, \nu_1).$$

The element $d(\Phi, \nu_1)$ is our representative for $d \in C_2/C_1$; it is defined up to $C_1(\Phi, \nu_1)$. In particular,

$$[d(\Phi, \nu_1)] = d(\Phi, \nu_1) \cdot C_1(\Phi, \nu_1).$$

Note that $d(\Phi, \nu_1) \cdot \nu_1 \in \mathfrak{a}'^*_-(C_1)$.

With the $d(\Phi, \nu_1)$ chosen, we are now ready to prove the following:

Proposition 4.5. *Let π_1 be an irreducible admissible representation of G_1, (P_1, ν_1, τ_1) Langlands data for G_1 such that $\pi_1 = L_1(P_1, \nu_1, \tau_1)$. If $d \in C_2/C_1$, then $(d(\Phi, \nu_1) \cdot P_1, d(\Phi, \nu_1) \cdot \nu_1, d(\Phi, \nu_1) \cdot \tau_1)$ is Langlands data for G_1, and we have*

$$d \cdot \pi_1 = L_1(d(\Phi, \nu_1) \cdot P_1, d(\Phi, \nu_1) \cdot \nu_1, d(\Phi, \nu_1) \cdot \tau_1).$$

Remark 4.1. *The triple $(d(\Phi, \nu_1) \cdot P_1, d(\Phi, \nu_1) \cdot \nu_1, d(\Phi, \nu_1) \cdot \tau_1)$ does not depend on the choice of $d(\Phi, \nu_1)$ because $M_1 = M_{\Phi, C_1(\Phi, \nu_1)}$, ν_1 and τ_1 are $C_1(\Phi, \nu_1)$-invariant.*

Proof. First, we have to show that $(d(\Phi, \nu_1) \cdot P_1, d(\Phi, \nu_1) \cdot \nu_1, d(\Phi, \nu_1) \cdot \tau_1)$ is Langlands data for G_1. Our choice of $d(\Phi, \nu_1)$ ensures that conditions 1. and 2. in the definition of Langlands data are satisfied. For 3., write $\Phi' = d(\Phi, \nu_1) \cdot \Phi$ and $(P_1', \nu_1', \tau_1') = d(\Phi, \nu_1) \cdot (P_1, \nu_1, \tau_1)$. Now $d(\Phi, \nu_1) \cdot M_\Phi = M_{\Phi'}$. By the abelianness of C,

$$d(\Phi, \nu_1) \cdot C_1(\Phi, \nu_1) = C_1(\Phi, \nu_1)$$

and

$$\begin{aligned} &C_1(\Phi', \nu_1') = \\ =&\{c \in C_1 | \, c \cdot \Phi' = \Phi', \, c \cdot \nu_1' = \nu_1'\} \\ =&\{c \in C_1 | \, cd(\Phi, \nu_1) \cdot \Phi = d(\Phi, \nu_1) \cdot \Phi, \, cd(\Phi, \nu_1) \cdot \nu_1 = d(\Phi, \nu_1) \cdot \nu_1\} \\ =&\{c \in C_1 | \, d(\Phi, \nu_1)c \cdot \Phi = d(\Phi, \nu_1) \cdot \Phi, \, d(\Phi, \nu_1)c \cdot \nu_1 = d(\Phi, \nu_1) \cdot \nu_1\} \\ =&C_1(\Phi, \nu_1). \end{aligned}$$

Thus, $d(\Phi,\nu_1)\cdot C_1(\Phi,\nu_1)=C_1(\Phi,\nu_1)=C_1(\Phi',\nu_1')$. Therefore,

$$d(\Phi,\nu_1)\cdot M_{\Phi,C_1(\Phi,\nu_1)}=M_{\Phi',C_1(\Phi',\nu_1')},$$

as needed. Finally, that 4. holds follows from Proposition 2.3.

By the preceding lemma,

$$\begin{aligned} d\cdot\pi_1 &= d(\Phi,\nu_1)\cdot\pi_1\\ &\hookrightarrow d(\Phi,\nu_1)\cdot i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1)\\ &\cong i_{G_1,d(\Phi,\nu_1)\cdot M_1}(exp\,d(\Phi,\nu_1)\cdot\nu_1\otimes d(\Phi,\nu_1)\cdot\tau_1). \end{aligned}$$

Since $(d(\Phi,\nu_1)\cdot P_1, d(\Phi,\nu_1)\cdot\nu_1, d(\Phi,\nu_1)\cdot\tau_1)$ is Langlands data for G_1, the Langlands classification for G_1 tells us that $L_1(d(\Phi,\nu_1)\cdot P_1, d(\Phi,\nu_1)\cdot\nu_1, d(\Phi,\nu_1)\cdot\tau_1)$ is the unique irreducible subrepresentation of

$$i_{G_1,d(\Phi,\nu_1)\cdot M_1}(exp\,d(\Phi,\nu_1)\cdot\nu_1\otimes d(\Phi,\nu_1)\cdot\tau_1).$$

Thus,

$$d\cdot\pi_1=L_1(d(\Phi,\nu_1)\cdot P_1, d(\Phi,\nu_1)\cdot\nu_1, d(\Phi,\nu_1)\cdot\tau_1),$$

as needed. □

We now proceed to the proof of the Langlands classification. Let

$$Lang(G_1)\xrightarrow{L_1} Irr(G_1)$$

$$Lang(G_1)\xleftarrow{T_1} Irr(G_1)$$

be the maps corresponding to the Langlands classification for G_1, which holds by inductive hypothesis ("T" for "triple"). Our first step is to construct

$$T_2\colon Irr(G_2)\to Lang(G_2).$$

Let $\pi_2\in Irr(G_2)$. Take $\pi_1\in Irr(G_1)$ such that π_1 appears in $r_{G_1,G_2}(\pi_2)$. Write $\pi_1=L_1(P_1,\nu_1,\tau_1)$. We define $T_2(\pi_2)$ in four cases. We also show that if $T_2(\pi_2)=(P_2,\nu_2,\tau_2)$, then π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2)$. Write $P_1^0=P_\Phi$.

Case 1: $C_2(\Phi)=C_1(\Phi)$.

In this case, it follows from the preceding proposition and Lemma 2.1. that, writing $D=C_2/C_1$,

$$r_{G_1,G_2}(\pi_2)=\bigoplus_{d\in D} L_1(d(\Phi,\nu_1)\cdot P_1, d(\Phi,\nu_1)\cdot\nu_1, d(\Phi,\nu_1)\cdot\tau_1).$$

(Note that $C_2(\Phi) = C_1(\Phi)$ implies the $d(\Phi,\nu_1)\cdot P_1$ are all different, so the representations in the right-hand side are inequivalent.) Further, for any $d \in D$,

$$\pi_2 \cong i_{G_2,G_1}(L_1(d(\Phi,\nu_1)\cdot P_1, d(\Phi,\nu_1)\cdot\nu_1, d(\Phi,\nu_1)\cdot\tau_1)).$$

We claim there is a unique $d \in D$ having $d(\Phi,\nu_1)\cdot\Phi \in X_{C_2}$. First, since

$$C_2\cdot\Phi = \bigcup_{d\in D} d(\Phi,\nu_1)C_1\cdot\Phi$$

and the fact that $d(\Phi,\nu_1)\cdot\Phi$ is maximal in $dC_1\cdot\Phi$ (by construction of $d(\Phi,\nu_1)$), such a $d \in D$ exists. For uniqueness, observe that if $d(\Phi,\nu_1)\cdot\Phi = d'(\Phi,\nu_1)\cdot\Phi$, then $d(\Phi,\nu_1)^{-1}d'(\Phi,\nu_1) \in C_2(\Phi) = C_1(\Phi)$, contradicting $d \neq d'$ in D. Write d_2 for this particular element of D. Set

$$T_2(\pi_2) = (d_2(\Phi,\nu_1)\cdot P_1, d_2(\Phi,\nu_1)\cdot\nu_1, d_2(\Phi,\nu_1)\cdot\tau_1) = (P_2,\nu_2,\tau_2).$$

Of course, we have to check that $(P_2,\nu_2,\tau_2) \in Lang(G_2)$.

First, d_2 was chosen so that $d_2(\Phi,\nu_1)\cdot\Phi \in X_{C_2}$, so condition 1. in the definition of Langlands data holds. By the preceding proposition, $(P_2,\nu_2,\tau_2) \in Lang(G_1)$. Since $C_2(\Phi) = C_1(\Phi)$, condition 2. in the definition of Langlands data is the same whether we want to view $(P_2,\nu_2,\tau_2) \in Lang(G_1)$ or $Lang(G_2)$. Thus 2. holds. For 3., observe that $C_2(d_2(\Phi,\nu_1)\cdot\Phi) = C_2(\Phi) = C_1(\Phi)$. Therefore,

$$\begin{aligned}
&C_2(d_2(\Phi,\nu_1)\cdot\Phi, d_2(\Phi,\nu_1)\cdot\nu_1)\\
&\qquad = \{c \in C_2(d_2(\Phi,\nu_1)\cdot\Phi) |\ cd_2(\Phi,\nu_1)\cdot\nu_1 = d_2(\Phi,\nu_1)\cdot\nu_1\}\\
&\qquad = \{c \in C_2(\Phi) |\ d_2(\Phi,\nu_1)c\cdot\nu_1 = d_2(\Phi,\nu_1)\cdot\nu_1\}\\
&\qquad = \{c \in C_1(\Phi) |\ c\cdot\nu_1 = \nu_1\}\\
&\qquad = C_1(\Phi,\nu_1).
\end{aligned}$$

Hence,

$$\begin{aligned}
d_2(\Phi,\nu_1)\cdot M_{\Phi,C_1(\Phi,\nu_1)} &= M_{d_2(\Phi,\nu_1)\cdot\Phi,C_1(\Phi,\nu_1)}\\
&= M_{d_2(\Phi,\nu_1)\cdot\Phi,C_2(d_2(\Phi,\nu_1)\cdot\Phi,d_2(\Phi,\nu_1)\cdot\nu_1)},
\end{aligned}$$

as needed. Thus 3. holds. Finally, that condition 4. holds follows from an argument like that for Proposition 2.4 (since $M_\Phi \neq M_{d(\Phi,\nu_1)\cdot\Phi}$, Proposition 2.4 is not quite enough). Therefore, we have $(P_2,\nu_2,\tau_2) \in Lang(G_2)$.

We now argue that π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$. Suppose π_2' is an irreducible representation which appears as a subrepresentation in $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$. Then,

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{G_2}(\pi_2', i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)) \\ &\cong \mathrm{Hom}_{G_1}(r_{G_1,G_2}(\pi_2'), i_{G_1,M_2}(exp\,\nu_2 \otimes \tau_2)). \end{aligned}$$

Now, $r_{G_1,G_2}(\pi_2') = \pi_1'$ or $\bigoplus d \cdot \pi_1'$. In either case, the Langlands classification for G_1 tells that we must have $d \cdot \pi_1' = L_1(P_2, \nu_2, \tau_2)$ for some d. Therefore, $\pi_2' \cong \pi_2$. In particular, only π_2 can appear as a subrepresentation. Further, the Langlands classification for G_1 also implies that

$$dim\,\mathrm{Hom}_{G_1}(r_{G_1,G_2}(\pi_2), i_{G_1,M_2}(exp\,\nu_2 \otimes \tau_2)) = 1,$$

so π_2 can appear only once as a subrepresentation. Thus, π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$.

Case 2: $C_2(\Phi) \neq C_1(\Phi)$ but $C_2(\Phi, \nu_1) = C_1(\Phi, \nu_1)$.

As in Case 1, we also have

$$r_{G_1,G_2}(\pi_2) = \bigoplus_{d \in D} L_1(d(\Phi, \nu_1) \cdot P_1, d(\Phi, \nu_1) \cdot \nu_1, d(\Phi, \nu_1) \cdot \tau_1)$$

and

$$\pi_2 \cong i_{G_2,G_1} L_1(d(\Phi, \nu_1) \cdot P_1, d(\Phi, \nu_1) \cdot \nu_1, d(\Phi, \nu_1) \cdot \tau_1),$$

for any $d \in D$. (Note that $C_2(\Phi, \nu_1) = C_1(\Phi, \nu_1)$ ensures that the $d(\Phi, \nu_1) \cdot (P_1, \nu_1, \tau_1)$ are distinct, so $r_{G_1,G_2}(\pi_2)$ decomposes into inequivalent representations as indicated.)

Observe that by Lemma 4.3, $C_2(\Phi)/C_1(\Phi) \cong C_2/C_1$. An easy check tells us if $d_1, \dots, d_p$ are representatives for $C_2(\Phi)/C_1(\Phi)$, they are also representatives for C_2/C_1. Now,

$$d_i C_1 \cdot \Phi = C_1 d_i \cdot \Phi = C_1 \cdot \Phi.$$

Since Φ is maximal in $C_1 \cdot \Phi$ (by condition 1. in the definition of Langlands data), we see that Φ is maximal in $d_i C_1 \cdot \Phi$. Consequently, Φ is maximal in $d(\Phi, \nu_1) C_1 \cdot \Phi$, so we must have $d(\Phi, \nu_1) \cdot \Phi = \Phi$ for any $d \in D$. Therefore, $d(\Phi, \nu_1) \cdot P_1 = P_1$ for any $d \in D$. Now, choose $d_2 \in D$ such that $d_2(\Phi, \nu_1) \cdot \nu_1$ is maximal among $\{d(\Phi, \nu_1) \cdot \nu_1\}_{d \in D}$, noting that $C_2(\Phi,\nu_1)=C_1(\Phi,\nu_1)$ tells us these will be distinct. We set

$$T_2(\pi_2)=(P_1,d_2(\Phi,\nu_1)\cdot\nu_1,d_2(\Phi,\nu_1)\cdot\tau_1)=(P_2,\nu_2,\tau_2).$$

We need to check $(P_2,\nu_2,\tau_2)\in Lang(G_2)$.

First, from above, we know that Φ is maximal in $dC_1\cdot\Phi$ for all $d\in D$. Therefore, Φ is maximal in $C_2\cdot\Phi$, i.e., $\Phi\in X_{C_2}$. Thus condition 1. in the definition of Langlands data is satisfied. The choice of d_2 and construction of $d_2(\Phi,\nu_1)$ ensures that condition 2. is satisfied. Condition 3. is just the fact $C_2(\Phi,\nu_2)=C_2(\Phi,\nu_1)=C_1(\Phi,\nu_1)$ (since $P_2=P_1$). Condition 4. is (again) a consequence of Proposition 2.4. Thus, $(P_2,\nu_2,\tau_2)\in Lang(G_2)$.

The argument that π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2)$ is essentially the same as in Case 1.

Case 3: $C_2(\Phi)\neq C_1(\Phi)$, $C_2(\Phi,\nu_1)\neq C_1(\Phi,\nu_1)$ but $C_2(\Phi,\nu_1,\tau_1)=C_1(\Phi,\nu_1,\tau_1)$.

Again, we have

$$r_{G_1,G_2}(\pi_2)=\bigoplus_{d\in D}L_1(d(\Phi,\nu_1)\cdot P_1,d(\Phi,\nu_1)\cdot\nu_1,d(\Phi,\nu_1)\cdot\tau_1)$$

and

$$\pi_2\cong i_{G_2,G_1}L_1(d(\Phi,\nu_1)\cdot P_1,d(\Phi,\nu_1)\cdot\nu_1,d(\Phi,\nu_1)\cdot\tau_1).$$

However, $T_2(\pi_2)$ will not be just a conjugate of (P_1,ν_1,τ_1) in this case.

First, we observe that, as in Case 2, $C_2(\Phi)\neq C_1(\Phi)$ implies $d(\Phi,\nu_1)\cdot\Phi=\Phi$ for all $d\in D$. Similarly, since $C_2(\Phi,\nu_1)\neq C_1(\Phi,\nu_1)$, we may also deduce that $d(\Phi,\nu_1)\cdot\nu_1=\nu_1$ for all $d\in D$. However, since $C_2(\Phi,\nu_1,\tau_1)=C_1(\Phi,\nu_1,\tau_1)$, we get that $\{d(\Phi,\nu_1)\cdot\tau_1\}_{d\in D}$ are inequivalent. (Note that this tells us $r_{G_1,G_2}(\pi_2)$ decomposes into inequivalent representations as indicated above.)

Let $P_2=P_{\Phi,C_2(\Phi,\nu_1)}$ and $\nu_2=\nu_1$. Since $\{d(\Phi,\nu_1)\cdot\tau_1\}_{d\in D}$ are inequivalent, we get

$$\tau_2=i_{M_2,M_1}(\tau_1)$$

is irreducible. We take

$$T_2(\pi_2)=(P_2,\nu_2,\tau_2).$$

Again, we must show that we actually have $(P_2, \nu_2, \tau_2) \in Lang(G_2)$.

First, as in Case 2, the fact that $C_2(\Phi) \neq C_1(\Phi)$ gives $\Phi \in X_{C_2}$, so condition 1. in the definition of Langlands data is satisfied. An argument very similar to the argument that $C_2(\Phi) \neq C_1(\Phi) \Rightarrow \Phi \in X_{C_2}$ tells us $C_2(\Phi, \nu_1) \neq C_1(\Phi, \nu_1) \Rightarrow \nu_1 \in \mathfrak{a}^*_-(C_2)$. Therefore, condition 2. is satisfied. Our choice of P_2 ensures 3. holds; 4. follows from the definition of tempered representations we are using. Therefore, $(P_2, \nu_2, \tau_2) \in Lang(G_2)$.

We now need to show that π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\, \nu_2 \otimes \tau_2)$. Since $i_{M_2,M_1}(\tau_1)$ is irreducible, we have $i_{G_2,M_2}(exp\, \nu_2 \otimes \tau_2) \cong i_{G_2,M_1}(exp\, \nu_2 \otimes \tau_1)$. Therefore, if π_2' is an irreducible subrepresentation of $i_{G_2,M_2}(exp\, \nu_2 \otimes \tau_2)$, we have

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_2}(exp\, \nu_2 \otimes \tau_2)) \\ &\cong \mathrm{Hom}_{G_1}(r_{G_1,G_2}(\pi_2'),\, i_{G_1,M_1}(exp\, \nu_1 \otimes \tau_1)). \end{aligned}$$

Now, either $r_{G_1,G_2}\pi_2' = \pi_1'$ or $\bigoplus d \cdot \pi_1'$. In either case, the Langlands classification for G_1 tells us we must have $d \cdot \pi_1' = L_1(P_1, \nu_1, \tau_1)$ for some d. Therefore, $\pi_2' \cong \pi_2$, so only π_2 can appear as a subrepresentation. Further, the Langlands classification for G_1 also tells us that

$$dim\, \mathrm{Hom}_{G_1}(r_{G_1,G_2}(\pi_2),\, i_{G_1,M_1}(exp\, \nu_1 \otimes \tau_1)) = 1,$$

so π_2 can appear only once as a subrepresentation. Thus, π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\, \nu_2 \otimes \tau_2)$, as needed.

Case 4: $C_2(\Phi) \neq C_1(\Phi)$, $C_2(\Phi, \nu_1) \neq C_1(\Phi, \nu_1)$, $C_2(\Phi, \nu_1, \tau_1) \neq C_1(\Phi, \nu_1, \tau_1)$.

As in Case 2, $C_2(\Phi) \neq C_1(\Phi)$ implies $d(\Phi, \nu_1) \cdot \Phi = \Phi$ for all $d \in D$. As in Case 3, $C_2(\Phi, \nu_1) \neq C_1(\Phi, \nu_1)$ implies $d(\Phi, \nu_1) \cdot \nu_1 = \nu_1$ for all $d \in D$. We claim that $C_2(\Phi, \nu_1, \tau_1) \neq C_1(\Phi, \nu_1, \tau_1)$ implies $d(\Phi, \nu_1) \cdot \tau_1 \cong \tau_1$ for all $d \in D$. Let $P_2 = P_{\Phi, C_2(\Phi, \nu_1)}$. Then

$$M_2/M_1 \cong C_2(\Phi, \nu_1)/C_1(\Phi, \nu_1) \cong C_2(\Phi, \nu_1, \tau_1)/C_1(\Phi, \nu_1, \tau_1)$$

(all three $\cong D$). Note that since $C_1(\Phi, \nu_1)$ acts trivially on $Irr(M_1)$, we get an action of $C_2(\Phi, \nu_1)/C_1(\Phi, \nu_1)$ on $Irr(M_1)$. Further, since $C_2(\Phi, \nu_1)/C_1(\Phi, \nu_1) \cong C_2(\Phi, \nu_1, \tau_1)/C_1(\Phi, \nu_1, \tau_1)$, it is easy to check that we may choose representatives for $C_2(\Phi, \nu_1)/C_1(\Phi, \nu_1)$ from $C_2(\Phi, \nu_1, \tau_1)$. Therefore, we see that the action of $C_2(\Phi, \nu_1)/C_1(\Phi, \nu_1)$ on τ_1 is trivial. The claim follows.

Observe that we now know $d(\Phi,\nu_1)\cdot(P_1,\nu_1,\tau_1)=(P_1,\nu_1,\tau_1)$ for all $d\in D$. Thus, from Lemma 2.1 and Proposition 4.5, we get

$$r_{G_1,G_2}(\pi_2)\cong L_1(P_1,\nu_1,\tau_1)=\pi_1$$

and

$$i_{G_2,G_1}L_1(P_1,\nu_1,\tau_1)\cong\bigoplus_{\chi\in\hat{D}}\chi\otimes\pi_2.$$

First, let $P_2=P_{\Phi,C_2(\Phi,\nu_1)}$ and $\nu_2=\nu_1$. Since $d(\Phi,\nu_1)\cdot\tau_1\cong\tau_1$ for all $d\in D$, it follows from Lemma 2.1 that

$$i_{M_2,M_1}(\tau_1)\cong\bigoplus_{\chi\in\hat{D}}\chi\otimes\tau_2,$$

for a fixed irreducible τ_2 appearing in $i_{M_2,M_1}(\tau_1)$. Observe that

$$\mathrm{Hom}_{G_2}(\pi_2,\, i_{G_2,M_1}(exp\,\nu_1\otimes\tau_1))\cong\mathrm{Hom}_{G_1}(\pi_1,\, i_{G_1,M_1}(exp\,\nu_1\otimes\tau_1))$$

is one-dimensional by the Langlands classification for G_1. On the other hand, by Frobenius reciprocity,

$$\begin{aligned}\mathrm{Hom}_{G_2}&(\pi_2,\, i_{G_2,M_1}(exp\,\nu_1\otimes\tau_1))\\ &\cong\mathrm{Hom}_{M_2}(r_{M_2,G_2}(\pi_2),\, exp\,\nu_1\otimes i_{M_2,M_1}(\tau_1))\\ &\cong\mathrm{Hom}_{M_2}(r_{M_2,G_2}(\pi_2),\, exp\,\nu_1\otimes(\bigoplus\chi\otimes\tau_2))\end{aligned}$$

(using induction in stages). Without loss of generality, we may choose τ_2 to be the component of $i_{M_2,M_1}(\tau_1)$ which has

$$\mathrm{Hom}_{M_2}(r_{M_2,G_2}(\pi_2),\, exp\,\nu_1\otimes\tau_2)\neq 0.$$

Then, we set

$$T_2(\pi_2)=(P_2,\nu_2,\tau_2).$$

Again, we need to check that $(P_2,\nu_2,\tau_2)\in Lang(G_2)$.

First, as in Case 3, $C_2(\Phi)\neq C_1(\Phi)$, $C_2(\Phi,\nu_1)\neq C_1(\Phi,\nu_1)$ ensure that $\Phi\in X_{C_2}$ and $\nu_2=\nu_1\in\mathfrak{a}^*_-(C_2)$. Thus, conditions 1. and 2. in the definition of Langlands data hold. Condition 3. follows immediately from $\nu_2=\nu_1$ and our choice of P_2. Condition 4. follows from Definition 2.5.

We now need to show that π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2)$. Suppose $\pi_2'\in Irr(G)$. From the

Langlands classification for G_1 and

$$\begin{aligned}\mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_1}(exp\,\nu_1 \otimes \tau_1))\\ \cong \mathrm{Hom}_{G_1}(r_{G_1,G_2}(\pi_2'),\, i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1)),\end{aligned}$$

we see that π_2' is not a subrepresentation of $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2) \hookrightarrow i_{G_2,M_1}(exp\,\nu_1 \otimes \tau_1)$ unless $\pi_2' = \chi \otimes \pi_2$. For such a π_2', we have

$$\begin{aligned}\mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2))\\ \cong \mathrm{Hom}_{M_2}(r_{M_2,G_2}(\pi_2'),\, exp\,\nu_2 \otimes \tau_2),\end{aligned}$$

is nonzero only when $\pi_2' \cong \pi_2$. Further, when $\pi_2' \cong \pi_2$ these are one-dimensional spaces, so π_2 appears only once as a subrepresentation. Thus, π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$, as needed.

At this point, we have constructed a map T_2: $Irr(G_2) \to Lang(G_2)$. Further, we have shown that if $T_2(\pi_2) = (P_2, \nu_2, \tau_2)$, then π_2 is the unique irreducible subrepresentation of $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$. Note that this implies T_2 is injective. To finish the proof of the theorem, it suffices to prove the following: if $(P_2, \nu_2, \tau_2) \in Lang(G_2)$, then $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$ has a unique irreducible subrepresentation. We can then define $L_2(P_2, \nu_2, \tau_2)$ to be that irreducible subrepresentation and it will follow easily that $T_2 \circ L_2 = id_{Lang(G_2)}$ and $L_2 \circ T_2 = id_{Irr(G_2)}$.

Let $(P_2, \nu_2, \tau_2) \in Lang(G_2)$. Write $P_1 = P_2 \cap G_1$. The argument that $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$ has a unique irreducible subrepresentation may be done in three cases:

1. $P_2 = P_1$.
2. $P_2 \neq P_1$ and $r_{M_2,M_1}(\tau_2)$ reducible.
3. $P_2 \neq P_1$ and $r_{M_2,M_1}(\tau_2)$ irreducible.

We remark that the arguments used below are, in some cases, quite similar to arguments used earlier.

Case 1: $P_2 = P_1$.

In this case, we also have $(P_2, \nu_2, \tau_2) \in Lang(G_1)$. Write

$$(P_1, \nu_1, \tau_1) = (P_2, \nu_2, \tau_2)$$

when considering it this way. By condition 3. in the definition of Langlands data and $M_2 = M_1$, we see that $C_2(\Phi, \nu_1) = C_1(\Phi, \nu_1)$. As in Cases 1. and 2. above (depending on whether $C_2(\Phi) = C_1(\Phi)$), this tells us that $d(\Phi, \nu_1) \cdot (P_1, \nu_1, \tau_1) \neq (P_1, \nu_1, \tau_1)$ for all $d \in D$ with $d \neq 1$. By Lemma 2.1. and Proposition 4.5, this implies

$$\pi_2 = i_{G_2,G_1}(L_1(P_1, \nu_1, \tau_1))$$

is irreducible. Now, for $\pi_2' \in Irr(G_2)$,

$$\begin{aligned} &\mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_2}(exp\, \nu_2 \otimes \tau_2)) \\ &\qquad \cong \mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,G_1} \circ i_{G_1,M_1}(exp\, \nu_1 \otimes \tau_1)) \\ &\qquad \cong \mathrm{Hom}_{G_1}(r_{G_1,G_2}(\pi_2'),\, i_{G_1,M_1}(exp\, \nu_1 \otimes \tau_1)) \end{aligned}$$

is one-dimensional if $L_1(P_1, \nu_1, \tau_1) \hookrightarrow r_{G_1,G_2}(\pi_2')$ and zero-dimensional otherwise (by the Langlands classification and the fact that $r_{G_1,G_2}(\pi_2')$ decomposes as a direct sum of inequivalent irreducible subrepresentations (possibly one)). That is, it is one-dimensional for $\pi_2' \cong \pi_2$ and zero-dimensional otherwise. Thus, $i_{G_2,M_2}(exp\, \nu_2 \otimes \tau_2)$ has a unique irreducible subrepresentation.

Case 2: $P_2 \neq P_1$ and $r_{M_2,M_1}(\tau_2)$ reducible.

Write

$$r_{M_2,M_1}(\tau_2) = \bigoplus_{d \in D} d \cdot \tau_1,$$

where τ_1 is an irreducible representation of M_1. If $\nu_1 = \nu_2$, then $(P_1, \nu_1, \tau_1) \in Lang(G_1)$. We note that $P_2 \neq P_1$ and condition 3. in the definition of Langlands data tells us $C_2(\Phi, \nu_1) \neq C_1(\Phi, \nu_1)$. This implies $C_2(\Phi) \neq C_1(\Phi)$ (the representatives for $C_2(\Phi, \nu_1)/C_1(\Phi, \nu_1)$) also serve as representatives for $C_2(\Phi)/C_1(\Phi)$). Therefore,

$$d(\Phi, \nu_1) \cdot (P_1, \nu_1, \tau_1) = (P_1, \nu_1, d(\Phi, \nu_1) \cdot \tau_1) \neq (P_1, \nu_1, \tau_1)$$

for $d \in D$ with $d \neq 1$. By Lemma 2.1. and Proposition 4.5, this implies

$$\pi_2 = i_{G_2,G_1}(L_1(P_1, \nu_1, \tau_1))$$

is irreducible.

In this case, $\tau_2 = i_{M_2,M_1}(\tau_1)$. Therefore, for π_2' irreducible,

$$\begin{aligned}\mathrm{Hom}_{G_2}(\pi_2',\, & i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)) \\ &\cong \mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_1}(exp\,\nu_1 \otimes \tau_1)) \\ &\cong \mathrm{Hom}_{G_1}(r_{G_1,G_2}(\pi_2'),\, i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1)).\end{aligned}$$

By the Langlands classification for G_1 (and the possibilities for $r_{G_1,G_2}(\pi_2')$ implied by Lemma 2.1.), this is one-dimensional in the case when $L_1(P_1,\nu_1,\tau_1) \hookrightarrow r_{G_1,G_2}(\pi_2')$ and zero-dimensional otherwise. In particular, it is one-dimensional for $\pi_2' \cong \pi_2$ and zero-dimensional otherwise. Thus, $i_{G_2,M_2}(exp\,\nu_2 \otimes \tau_2)$ has a unique irreducible subrepresentation.

Case 3: $P_2 \neq P_1$ and $r_{M_2,M_1}(\tau_2)$ irreducible.

Write

$$r_{M_2,M_1}(\tau_2) = \tau_1,$$

with τ_1 irreducible. If $\nu_1 = \nu_2$, then $(P_1,\nu_1,\tau_1) \in Lang(G_1)$. As in Case 2, $P_2 \neq P_1$ implies $C_2(\Phi,\nu_1) \neq C_1(\Phi,\nu_1)$, and therefore $C_2(\Phi) \neq C_1(\Phi)$. Further, the irreducibility of $r_{M_2,M_1}(\tau_2)$ then implies

$$d(\Phi,\nu_1)\cdot(P_1,\nu_1,\tau_1) = (P_1,\nu_1,\tau_1)$$

for all $d \in D$. (We note that we may also conclude $C_2(\Phi,\nu_1,\tau_1) = C_1(\Phi,\nu_1,\tau_1)$.) In particular, by Lemma 2.1. and Proposition 4.5,

$$i_{G_2,G_1}L_1(P_1,\nu_1,\tau_1) = \bigoplus_{\chi\in\hat{D}} \chi \otimes \pi_2,$$

where π_2 is any fixed component of $i_{G_2,G_1}L_1(P_1,\nu_1,\tau_1) = i_{G_2,G_1}(\pi_1)$. Observe that for $\pi_2' \in Irr(G_2)$

$$\begin{aligned}\mathrm{Hom}_{G_2}(\pi_2',\, & i_{G_2,M_1}(exp\,\nu_1 \otimes \tau_1)) \\ &\cong \mathrm{Hom}_{G_1}(r_{G_1,G_2}(\pi_2'),\, i_{G_1,M_1}(exp\,\nu_1 \otimes \tau_1))\end{aligned}$$

is one-dimensional if $r_{G_1,G_2}(\pi_2') = \pi_1$ and zero-dimensional otherwise (by the Langlands classification for G_1). That is, it is one-dimensional if $\pi_2' \in \{\chi \otimes \pi_2\}_{\chi\in\hat{D}}$ and zero-dimensional otherwise. On the other hand, since $i_{M_2,M_1}(\tau_1) \cong \bigoplus_{\chi\in\hat{D}} \chi \otimes \tau_2$, $(D \cong M_2/M_1 \cong G_2/G_1)$, we have

$$\begin{aligned}&\mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_1}(exp\,\nu_1\otimes\tau_1))\\ &\qquad\cong \mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_2}(exp\,\nu_2\otimes i_{M_2,M_1}(\tau_1)))\\ &\qquad\cong \mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_2}(exp\,\nu_2\otimes(\bigoplus_{\chi\in\hat{D}}\chi\otimes\tau_2)).\end{aligned}$$

Take $\pi_2' \in \{\chi\otimes\pi_2\}_{\chi\in\hat{D}}$. By one-dimensionality, there is a unique $\tau_2'\in\{\chi\otimes\tau_2\}_{\chi\in\hat{D}}$ such that $\mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2'))$ is nonzero; by cardinality considerations, $\pi_2'\leftrightarrow\tau_2'$ is bijective. We choose π_2 to be the component of $i_{G_2,G_1}(\pi_1)$ corresponding to τ_2. This implies that $\mathrm{Hom}_{G_2}(\pi_2',\, i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2))$ is one-dimensional if $\pi_2'\cong\pi_2$ and zero-dimensional otherwise. Therefore, $i_{G_2,M_2}(exp\,\nu_2\otimes\tau_2)$ has a unique irreducible subrepresentation (namely π_2).

This finishes the proof of the theorem. □

Remark 4.2. *The Langlands classification may also be formulated in the quotient setting. We close by doing this.*

First, for $P = MU$ a standard parabolic subgroup of G, let

$$\mathfrak{a}_+^* = \{x\in\mathfrak{a}^* \mid \langle x,\alpha\rangle > 0,\, \forall\,\alpha\in\Pi(P^0,A)\},$$

$$\mathfrak{a}_+^*(C) = \{x\in\mathfrak{a}_+^* \mid x\preceq c\cdot x,\, \forall\, c\in C(\Phi)\}.$$

We note that $\mathfrak{a}_+^* = -\mathfrak{a}_-^*$ and $\mathfrak{a}_+^*(C) = -\mathfrak{a}_-^*(C)$. A triple (P,ν,τ) is a set of Langlands data for the quotient setting of the Langlands classification if the following hold:

1. *$P = MU$ is a standard parabolic subgroup of G*
2. *$\nu\in\mathfrak{a}_+^*(C)$*
3. *$M = M_{\Phi,C(\Phi,\nu)}$*
4. *$\tau\in Irr(M)$ is tempered.*

In this case, we have a bijective correspondence

$$Irr(G)\longleftrightarrow Lang_{quot}(G).$$

If $\pi\in Irr(G)$ has Langlands quotient data (P,ν,τ), then π is the unique irreducible quotient of $Ind_P^G(exp\,\nu\otimes\tau)$.

It is not difficult to obtain the quotient version of the Langlands classification from the subrepresentation version. Let $\pi\in Irr(G)$. Write $\tilde{\pi} = L(P,\nu,\tau)$ (subrepresentation setting), where $\tilde{\pi}$ denotes the contragredient of π. By the contravariance of $\tilde{}$, the fact that $\tilde{\pi}$

is the unique irreducible subrepresentation of $Ind_P^G(exp\,\nu\otimes\tau)$ *tells us that* $\tilde{\tilde{\pi}}\cong\pi$ *is the unique irreducible quotient of* $(Ind_P^G(exp\,\nu\otimes\tau))\tilde{}\cong Ind_P^G(exp(-\nu)\otimes\tilde{\tau})$. *It is easy to check that* $(P,-\nu,\tilde{\tau})\in Lang_{quot}(G)$. *This argument may be reversed to establish the equivalence of the two formulations.*

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